Chapter 1

Exercise 1.1.i.i: Show that a morphism can have at most one inverse isomorphism.

Given $f : x \to y$ and $g, g' : y \to x$ with $fg = 1_y$, $gf = 1_x$, $fg' = 1_y$ and $g'f = 1_x$, then $g = 1_x g = g'fg = g'1_y = g'$

Exercise 1.1.i.ii: Consider a morphism $f : x \to y$. Show that if there exists a pair of morphisms $g, h : y \rightrightarrows x$ so that $gf = 1_x$ and $fh = 1_y$, then $g = h$ and f is an isomorphism.

Then $g = g1_y = gfh = 1_x h = h$ so that $fg = fh = 1_y$ and we already know that $gf = 1_x$ hence f is an isomorphism.

Exercise 1.1.1.iii: For any category C and any object $c \in \mathsf{C}$, show that:

(i) There is a category c/C whose objects are morphisms $f : c \to x$ with domain c and in which a morphism from $f : c \to x$ to $g : c \to y$ is a map $h : x \to y$ between the codomains so that the triangle

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

commutes, i.e., so that $g = hf$
Suppose $f : c \to x, g : c \to y, h : c \to z$ are objects of c/C and $\alpha : x \to y, \beta : y \to z$ are morphisms $f \to g$ and $g \to h$ in c/C. In that case we have $\alpha f = g$ and $\beta g = h$. Then define composition $\beta\alpha$ in c/C as composition in C. This is a morphism $f \to h$ in c/C because

$$(\beta\alpha)f = \beta(\alpha f) = \beta g = h$$

Associativity follows from associativity in C.
Define the identity 1_f for $f : c \to x$ as the identity 1_x in C. Then given $\alpha : f \to g$ ($\alpha : x \to y$ and $\alpha f = g$), we have $\alpha 1_f = \alpha 1_x = \alpha$ and $1_g \alpha = 1_y \alpha = \alpha$.

(ii) There is a category C/c whose objects are morphisms $f : x \to c$ with codomain c and in which a morphism from $f : x \to c$ to $g : y \to c$ is a map $h : x \to y$ between the codomains so that the triangle

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

commutes, i.e., so that $f = gh$.

Exercise 1.2.i: Show that $\mathsf{C}/c \cong (c/\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$. Defining C/c to be $(c/\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$, deduce Exercise 1.1.iii(ii) from Exercise 1.1.iii(i).

Say f is an object of $(c/\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$ which is, by definition, simply an object of $c/\mathsf{C}^{\mathrm{op}}$ which is a morphism $f^{\mathrm{op}} : c \to x$ in C^{op} which is simply a morphism $f : x \to c$ in C. This is the definition of objects in C/c.

Now say f and g are objects of $(c/\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$ which means they are morphisms $f^{\mathrm{op}} : c \to x$ and $g^{\mathrm{op}} : c \to y$ and say $\alpha^{\mathrm{op}} : f \to g$ is a morphism in $(c/\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$. This means that $\alpha : g^{\mathrm{op}} \to f^{\mathrm{op}}$ is a morphism in $c/\mathsf{C}^{\mathrm{op}}$. This means α is a morphism $\alpha^{\mathrm{op}} : y \to x$ in C^{op} such that $\alpha^{\mathrm{op}} g^{\mathrm{op}} = f^{\mathrm{op}}$. Then

$$\alpha^{\mathrm{op}} g^{\mathrm{op}} = f^{\mathrm{op}} \Leftrightarrow (g\alpha)^{\mathrm{op}} = f^{\mathrm{op}} \Leftrightarrow g\alpha = f$$

which means α is a morphism $f \to g$ in C/c.

We deduce that $\mathsf{C}/c \cong (c/\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$ is a category as follows: $c/\mathsf{C}^{\mathrm{op}}$ is a category because C is and $(c/\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$ is a category because $c/\mathsf{C}^{\mathrm{op}}$ is.

Exercise 1.2.ii:

(i) Show that a morphism $f : x \to y$ is a split epimorphism in a category $\mathbf{C}$ if and only if for all $c \in \mathsf{C}$, post-composition $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ defines a surjective function.

Proof: If f is a split epi then we have $f' : y \to x$ such that $ff' = 1_y$. Given $g : c \to y$ let $g' = f'g$ in which case post-composition gives $f_*(g') = fg' = ff'g = 1_y g = g$ so that f_* is a surjection.
In the other direction, if f_* is a surjection then $1_y : y \to y$ is in its image which is to say there exists $f' : y \to x$ such that $f_*(f') = ff' = 1_y$. Thus f is a split epi.

(ii) Argue by duality that f is a split monomorphism if and only if for all $c \in \mathsf{C}$, pre-composition $f^* : \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ defines a surjective function.

By definition, $f : x \to y$ is a split mono if and only if $f^{\mathrm{op}} : y \to x$ is a split epi in $\mathbf{C}$. This is the case if and only if post-composition $f_*^{\mathrm{op}} : \mathsf{C}^{\mathrm{op}}(c, y) \to \mathsf{C}^{\mathrm{op}}(c, x)$ is a surjection by the previous exercise. This is saying $f^{\mathrm{op}} g^{\mathrm{op}} = (gf)^{\mathrm{op}}$ is a surjection on morphisms $g'^{\mathrm{op}} : c \to x$ which is the same as pre-composition gf being a surjection to $g' : x \to c$.

Example 1.3.15: Every morphism $G/H \to G/K$ has the form $gH \mapsto g\gamma K$, where $\gamma \in G$ is an element so that $\gamma^{-1} H \gamma \subset K$.

Let $f : G/H \to G/K$ be such a morphism. Then $f(H) = \gamma K$ for some $\gamma \in G$. Then for any $h \in H$

$$\begin{aligned}
\gamma^{-1} h \gamma K &= \gamma^{-1} h f(H) \\
&= \gamma^{-1} f(hH) \quad \textit{Since } f \textit{ is } G\textit{-equivariant} \\
&= \gamma^{-1} f(H) \\
&= \gamma^{-1} \gamma K \\
&= K
\end{aligned}$$

Thus $\gamma^{-1} h \gamma \in K$.

Now given $\gamma \in G$ with the assumption that $\gamma^{-1} H \gamma \subset K$, define $f(gH) = g\gamma H$. This is well-defined since for another $g'H = gH$ (i.e. $g'^{-1} g \in H$), we have

$$
\begin{aligned}
g\gamma K &= g'g'^{-1}g\gamma K \\
&= g'\gamma\gamma^{-1}g'^{-1}g\gamma K \\
&= g'\gamma(\gamma^{-1}g'^{-1}g\gamma)K \\
&= g'\gamma K
\end{aligned}
$$

since by assumption $\gamma^{-1}g'^{-1}g\gamma \in K$ due to $g'^{-1}g \in H$.

Exercise 1.3.i: What is a functor between groups, regarded as on-object categories? A morphism of groups.

Exercise 1.3.ii: What is a functor between preorders, regarded as categories? An order-preserving function, i.e. if $a \le b$ then $fa \le fb$.

Exercise 1.3.iii: Find an example to show that the objects and morphisms in the image of a functor $F : C \to D$ do not necessarily define a subcategory of D.

Map the category

$$
a \xrightarrow{f} b \qquad\qquad c \xrightarrow{g} d
$$

to

$$
\begin{tikzcd}
Fa \arrow[r, "Ff"] & Fb = Fc \arrow[d, "Fg"] \\
& Fd
\end{tikzcd}
$$

This image does not contain the composite $Fg \cdot Ff$ and so is not a category.

Exercise 1.3.viii: Lemma 1.3.8 shows that functors preserve isomorphisms. Find an example to demonstrate that functors need not **reflect isomorphisms**: that is, find a functor $F : \mathsf{C} \to \mathsf{D}$ and a morphism f in C so that Ff is an isomorphism in D but f is not an isomorphism in C.

Map

$$
\cdot \xrightarrow{f} \cdot
$$

to

$$
\begin{tikzcd}
\cdot \arrow[r, shift left, "Ff=g"] & \cdot \arrow[l, shift left, "g^{-1}"]
\end{tikzcd}
$$

Exercise 1.4.i: Suppose $\alpha : F \Rightarrow G$ is a natural isomorphism. Show that the inverses of the component morphisms define the components of a natural isomorphism $\alpha^{-1} : G \Rightarrow F$.

For any $f : c \to c'$ in C we need to show that the diagram

$$
\begin{tikzcd}
Gc \arrow[r, "\alpha_c^{-1}"] \arrow[d, "Gf"] & Fc \arrow[d, "Ff"] \\
Gc' \arrow[r, "\alpha_{c'}^{-1}"] & Fd
\end{tikzcd}
$$

commutes.

This is true since

$$\begin{aligned}
Ff \cdot \alpha_c^{-1} &= \alpha_{c'}^{-1} \cdot \alpha_{c'} \cdot Ff \cdot \alpha_c^{-1} \\
&= \alpha_{c'}^{-1} \cdot (\alpha_{c'} \cdot Ff) \cdot \alpha_c^{-1} \\
&= \alpha_{c'}^{-1} \cdot (Gf \cdot \alpha_c) \cdot \alpha_c^{-1} \quad \textit{(Since } \alpha \textit{ is natural)} \\
&= \alpha_{c'}^{-1} \cdot Gf
\end{aligned}$$

Exercise 1.4.ii: What is a natural transformation between a parallel pair of functors between groups, regarded as categories?

Given the groups G and H and group homomorphisms $f, g : G \to H$, then elements $\gamma \in G, f(\gamma) \in H, g(\gamma) \in H$ correspond to arrows in the one object categories $\mathsf{B}G$ and $\mathsf{B}H$ and f and g correspond to functors $\mathsf{B}G \to \mathsf{B}H$. By definition, a natural transformation $\alpha : f \Rightarrow g$ must correspond to a single element $\alpha \in H$ such that the following diagram commutes

$$\begin{tikzcd}
\cdot \arrow[r, "\alpha"] \arrow[d, "f(\gamma)"'] & \cdot \arrow[d, "g(\gamma)"] \\
\cdot \arrow[r, "\alpha"] & \cdot
\end{tikzcd}$$

for any $\gamma \in G$.

Equivalently we have $\alpha f(\gamma)\alpha^{-1} = g(\gamma)$ which is to say that the composition of f with the inner automorphism ϕ_α defined by α is g. Thus natural transformations correspond to inner automorphisms that make the following diagram commute in the category Group.

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[dr, "g"'] & H \arrow[d, "\phi_\alpha"] \\
& H
\end{tikzcd}$$

Exercise 1.4.iii: What is a natural transformation between a parallel pair of functors between preorders, regarded as categories?

Given two functors f and g between preorders a natural transformation simply means that $f(x) \leq g(x)$ for all x in the domain.

Exercise 1.4.iv: In the notation of Example 1.4.7, prove that distinct parallel morphisms $f, g : c \rightrightarrows d$ define distinct natural transformations

$$f_*, g_* : \mathsf{C}(-, c) \Rightarrow \mathsf{C}(-, d) \quad \text{and} \quad f^*, g^* : \mathsf{C}(d, -) \Rightarrow \mathsf{C}(c, -)$$

by post- and pre-composition.

To show that f_* and g_* are distinct, they must have a distinct component at some particular object of C. By definition, the component of f_* at $b \in \mathsf{C}$ is the function that sends $h \in \mathsf{C}(b, c)$ to $fh \in \mathsf{C}(b, d)$. So take $c \in \mathsf{C}$ and notice that

$$f_{*c}(1_c) = f1_c = f \neq g = g1_c = g_{*c}(1_c)$$

showing that f_* and g_* are distinct transformations. The proof for f^* and g^* is similar.

Exercise 1.4.v: Recall the construction of the comma category for any pair of functors $F : \mathsf{D} \to \mathsf{C}$ and $G : \mathsf{E} \to \mathsf{C}$ described in Exercise 1.3.vi. From this data, construct a canonical natural transformation $\alpha : F\mathrm{dom} \Rightarrow G\mathrm{cod}$ between the functors that from the boundary of the square

$$\begin{tikzcd} F \downarrow G \arrow[r, "\mathrm{cod}"] \arrow[d, "\mathrm{dom}"'] \arrow[dr, phantom, "\overset{\alpha}{\nearrow}"] & \mathsf{E} \arrow[d, "G"] \\ \mathsf{D} \arrow[r, "F"'] & \mathsf{C} \end{tikzcd}$$

The comma category is composed of triples $(d \in \mathsf{D}, e \in \mathsf{E}, f : Fd \to Ge)$ with arrows tuples $(g : d \to d', h : e \to e')$ such that the diagram

$$\begin{tikzcd} Fd \arrow[r, "f"] \arrow[d, "Fg"] & Ge \arrow[d, "Fh"] \\ Fd' \arrow[r, "f'"] & Ge' \end{tikzcd} \tag{1}$$

commutes.

Since $\mathrm{dom}(d, e, f) = d$ and $\mathrm{cod}(d, e, f) = e$, diagram (1) can be rewritten

$$\begin{tikzcd} F \cdot \mathrm{dom}(d, e, f) \arrow[r, "f"] \arrow[d, "Fg"] & G \cdot \mathrm{cod}(d, e, f) \arrow[d, "Fh"] \\ F \cdot \mathrm{dom}(d', e', f') \arrow[r, "f'"] & G \cdot \mathrm{cod}(d', e', f') \end{tikzcd}$$

showing that the arrows f assemble into a natural transformation. Thus define the component of α at (d, e, f) to be $\alpha_{(d,e,f)} = f$.

Exercise 1.5.i: Prove Lemma 1.5.1: Fixing a parallel pair of functors $F, G : \mathsf{C} \rightrightarrows \mathsf{D}$, natural transformations $\alpha : F \Rightarrow G$ correspond bijectively to functors $H : \mathsf{C} \times \mathbb{2} \to \mathsf{D}$ such that H restricts along i_0 and i_1 to the functors F and G, i.e., so that

$$\begin{tikzcd} \mathsf{C} \arrow[r, "i_0"] \arrow[dr, "F"'] & \mathsf{C} \times \mathbb{2} \arrow[d, "H"] & \mathsf{C} \arrow[l, "i_1"'] \arrow[dl, "G"] \\ & \mathsf{D} & \end{tikzcd} \tag{2}$$

Given F, G and α, define H on objects by $H(c, 0) = Fc$ and $H(c, 1) = Gc$ and on arrows by $H(f, 1_0) = Ff$, $H(f, 1_1) = Gf$ and $H(f, f_{01})$ is the common composite $Gf \cdot \alpha_c = \alpha_{c'} \cdot Ff$ where f_{01} is the unique arrow $0 \to 1 \in \mathbb{2}$. Since $i_0(c) = (c, 0)$ and $i_0(f) = (f, 1_0)$ (and similarly for i_1), this satisfies diagram (2).

Given composition rules, there are four cases to consider to check if this is a functor:

$$(f, 1_0)(g, 1_0) \tag{1}$$
$$(f, 1_1)(g, 1_1) \tag{2}$$
$$(f, 1_1)(g, f_{01}) \tag{3}$$
$$(f, f_{01})(g, 1_0) \tag{4}$$

For case (1),

$$\begin{aligned} H((f,1_0)(g,1_0)) &= H(fg,1_0) \\ &= F(fg) \\ &= Ff \cdot Fg \\ &= H(f,1_0)H(g,1_0) \end{aligned}$$

and case (2) is similar. For cases (3) and (4),

$$\begin{aligned} H((f,1_1)(g,f_{01})) &= H(fg,f_{01}) \\ &= G(fg)\alpha_c \\ &= Gf \cdot Gg \cdot \alpha_c \\ &= H(f,1_1)H(g,f_{01}) \\ H((f,f_{01})(g,1_0)) &= H(fg,f_{01}) \\ &= \alpha_{c'}F(fg) \\ &= \alpha_{c'} \cdot Ff \cdot Fg \\ &= H(f,f_{01})H(g,1_0) \end{aligned}$$

Conversely, assume a functor $H : \mathsf{C} \times \mathbb{2} \to D$ such that diagram (2) commutes. Since, by the diagram, $H(c,1_0) = Fc$ and $H(c,1_1) = Gc$ and $(1_c,f_{01}) : (c,1_0) \to (c,1_1)$, we have $H(1_c,f_{01}) : Fc \to Gc$. Thus, define components $\alpha_c = H(1_c,f_{01})$. Then for $f : c \to c'$,

$$\begin{aligned} \alpha_{c'} \cdot Ff &= H(1_{c'},f_{01})H(f,1_0) \\ &= H(f,f_{01}) \\ &= H(f,1_1)H(1_c,f_{01}) \\ &= Gf \cdot \alpha_c \end{aligned}$$

so that the components assemble into a natural transformation $\alpha : F \Rightarrow G$.

Exercise 1.5.ii: Prove that Γ is equivalent to the opposite of the category Fin_* of finite pointed sets.

Define $F : \Gamma \to \mathsf{Fin}_*$ by sending $S \to (S \cup \{S\}, \{S\})$. Given a morphism $S \to T \in \Gamma$ corresponding to a function $\theta : S \to P(T)$, define the based function function $F\theta : T \cup \{T\} \to S \cup \{S\}$ as follows,

$$t \mapsto \begin{cases} s & \text{if } t \in \theta(s) \text{ for some } s \in S. \\ \{S\} & \text{otherwise.} \end{cases}$$

This is well-defined since the sets $\theta(s)$ for $s \in S$ are all disjoint.

The other functor $G : \mathsf{Fin}_* \to \Gamma$ is defined on objects by $(X,x) \mapsto X \setminus x$ and on arrows $f : (X,x) \to (Y,y)$ by sending $a \in Y \setminus y$ to the pre-image $f^{-1}(a) \subseteq X \setminus x$, thus a function $Gf : Y \setminus y \to P(X \setminus x)$. I'm not going to prove the natural isomorphism, it is basically the same as given in Example 1.5.6.

Exercise 1.5.iii: Prove Lemma 1.5.10: Any morphism $f : a \to b$ and fixed isomorphisms $a \cong a'$ and $b \cong b'$ determine a unique morphism $f' : a' \to b'$ so that any of—or, equivalently, all of—the following four diagrams commute:

$$\begin{tikzcd} a \arrow[d, "f"'] & a' \arrow[l, "\cong"'] \arrow[d, "f'"] \\ b \arrow[r, "\cong"'] & b' \end{tikzcd} \quad \begin{tikzcd} a \arrow[r, "\cong"] \arrow[d, "f"'] & a' \arrow[d, "f'"] \\ b \arrow[r, "\cong"'] & b' \end{tikzcd} \quad \begin{tikzcd} a \arrow[d, "f"'] & a' \arrow[l, "\cong"'] \arrow[d, "f'"] \\ b & b' \arrow[l, "\cong"] \end{tikzcd} \quad \begin{tikzcd} a \arrow[r, "\cong"] \arrow[d, "f"'] & a' \arrow[d, "f'"] \\ b & b' \arrow[l, "\cong"] \end{tikzcd}$$

Let $\alpha : a \xrightarrow{\cong} a'$ and $\beta : b \xrightarrow{\cong} b'$ be the isomorphisms and define $f' = \beta f \alpha^{-1}$. The four relations therefore are

$$\begin{aligned} f' &= \beta f \alpha^{-1} \\ f'\alpha &= \beta f \\ \beta^{-1} f' &= f \alpha^{-1} \\ \beta^{-1} f' \alpha &= f \end{aligned}$$

If any are true, then all are true simply by moving the isomorphisms around. If any of the relations are true for some other $g : a' \to b'$ then $g = \beta f \alpha^{-1} = f'$ so that the arrow is unique.

Exercise 1.5.iv: Show that a full and faithful functor $F : C \to D$ both **reflects** and **creates isomorphisms**. That is show:

(i) If f is a morphism in C so that Ff is an isomorphism in D, then f is an isomorphism.
(ii) If x and y are objects in C so that Fx and Fy are isomorphic in D, then x and y are isomorphic in C.

By Lemma 1.3.8, the converses of these statements hold for any functor.

(i) Since F is full and Ff is an isomorphism, there is some g so that $Fg = (Ff)^{-1}$. Then $F(gf) = Fg \cdot Ff = (Ff)^{-1}Ff = 1_{Fx}$. Since also $F1_x = 1_{Fx}$ and because F is faithful we must have that $gf = 1_x$. Similarly $fg = 1_y$ showing that $g = f^{-1}$.
(ii) Since F is full, there exists $f : x \to y$ such that $Ff : Fx \xrightarrow{\cong} Fy$ and $g : y \to x$ such that $Fg = (Ff)^{-1}$. Then $F(gf) = Fg \cdot Ff = 1_{Fx}$ whence $gf = 1_x$ by faithfulness. The other side follows by the same argument.

Exercise 1.5.v: Find an example to show that a faithful functor need not reflect isomorphisms.

$$x \xrightarrow{f} y \quad \mapsto \quad \begin{tikzcd} Fx \arrow[r, shift left, "Ff"] & Fy \arrow[l, shift left, "(Ff)^{-1}"] \end{tikzcd}$$

Exercise 1.5.vi: Just showing that a composite of natural isomorphisms is a natural isomorphism. This seems completely trivial to prove so I'm leaving it.

Exercise 1.5.vii: Let G be a connected groupoid and let G be the group of automorphisms of any of its objects. The inclusion $\mathsf{B}G \hookrightarrow \mathsf{G}$ defines an equivalence of categories. Construct an inverse equivalence $\mathsf{G} \to \mathsf{B}G$.

Let g be the image of $\mathsf{B}G$ in G. For each $h \in \mathsf{G}$ choose one particular isomorphism $\epsilon_h : g \to h$. For g, choose $\epsilon_g = 1_g$. Then to define the functor $\mathsf{G} \to \mathsf{B}G$, send each h to g and send any $f : h \to h'$ to $\epsilon_{h'}^{-1} f \epsilon_h : g \to g$.

$$\begin{tikzcd} g \arrow[r, "\epsilon_h"] \arrow[d, "\epsilon_{h'}^{-1} f \epsilon_h"'] & h \arrow[d, "f"] \\ g & h' \arrow[l, "\epsilon_{h'}^{-1}"] \end{tikzcd}$$

This is a functor since

$$\epsilon_{h''}^{-1} f' \epsilon_{h'} \epsilon_{h'}^{-1} f \epsilon_h = \epsilon_{h''}^{-1} f' f \epsilon_h$$

and

$$\epsilon_h^{-1} 1_h \epsilon_h = \epsilon_h^{-1} \epsilon_h = 1_g$$

Exercise 1.5.viii: Inverse equivalence $\mathsf{Proj}^{\mathsf{l}} \to \mathsf{Affine}$.

Given an affine plane A, add a point for every equivalence class of parallel lines and add incidence data where each line in that equivalence class intersects the corresponding point. Finally add a line consisting of all those points—with corresponding incidence data—and designate it to be the line at infinity.

Exercise 1.5.ix: Show that any category that is equivalent to a locally small category is locally small.

Assume we have the the functor $F : \mathsf{C} \to \mathsf{D}$ that is full, faithful and essentially surjective and that D is locally small. By faithfulness, if there is an arrow $f : x \to y$ such that $Ff = g$ for some $g : Fx \to Fy \in \mathsf{C}$, then that arrow is unique. By the axiom schema of replacement, there exists a set S of arrows $f : x \to y$ such that $Ff \in \mathsf{D}(Fx, Fy)$ since $\mathsf{D}(Fx, Fy)$ is a set. By fullness, this is all the arrows $f : x \to y$, that is, $S = \mathsf{C}(x, y)$.

Exercise 1.6.i: Show that any map from a terminal object in a category to an initial one is an isomorphism. An object that is both initial and terminal is called a **zero object**.

Assume we have $f : t \to i$. Since i is initial (and equally because t is terminal) there is a unique arrow $g : i \to t$. Then $gf : t \to t$ must be 1_t since, by uniqueness of arrows to t, there is only one arrow $t \to t$. Similarly $gf : i \to i$ is 1_i since by uniqueness of arrows from i there is only one arrow $i \to i$.

Exercise 1.6.ii: Show that any two terminal objects in a category are connected by a unique isomorphism.

Assuming that t_1 and t_2 are both terminal, we have unique arrows $f_1 : t_2 \to t_1$ because t_1 is terminal and $f_2 : t_1 \to t_2$ because t_2 is. Then $f_1 f_2$ must be 1_{t_1} becuase t_1 is terminal (uniqueness of arrows to t_1) and similarly $f_2 f_1 = 1_{t_2}$.

Exercise 1.6.iii: Show that any faithful functor reflects monomorphisms. That is, if $F : \mathsf{C} \to \mathsf{D}$ is faithful, prove that if Ff is a monomorphism in D, then f is a monomorphism in C. Argue by duality that faithful functors also reflect epimorphisms. Conclude that in any concrete category, any morphism that defines an injection of underlying sets is a monomorphism and any morphism that defines a surjection of underlying sets is an epimorphism.

Assume that $fg = fh$ for some arrows g, h. Then $Ff \cdot Fg = Ff \cdot Fh$ wherefore $Fg = Fh$ since Ff is monic. Because F is faithful, $g = h$. The functor $F : \mathsf{C}^{op} \to \mathsf{D}^{op}$ defined by F is also faithful whereby F also preserves epimorphisms by duality.

Since monics and epics in Set are injections and surjections respectively, these reflect into monics and epics in the corresponding concrete category.

Exercise 1.6.iv: Fin an example to show that a faithful functor need not preserve epimorphisms. Argue by duality, or by another counterexample, that a faithful functor need not perserve monomorphisms.

Send

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[dr, "gf"'] & \cdot \arrow[d, "g"] \\ & \cdot \end{tikzcd}$$

to

$$\begin{tikzcd} \cdot \arrow[r, "Ff"] \arrow[dr, "F(gf)=hFf"'] & \cdot \arrow[d, shift right] \arrow[d, shift left, "Fg\neq h"] \\ & \cdot \end{tikzcd}$$

A concrete realization of this functor is:

$$\begin{tikzcd} \{a\} \arrow[r] \arrow[dr] & \{b\} \arrow[d] \\ & \{c\} \end{tikzcd}$$

to

$$\begin{tikzcd} \{a\} \arrow[r, "f"] \arrow[dr, "h"'] & \{b, b_1\} \arrow[d, shift right, "g_1"'] \arrow[d, shift left, "g_2"] \\ & \{c, c_1\} \end{tikzcd}$$

where

$$f(a) = b$$

$$g_1(b) = g_1(b_1) = c$$

$$g_2(b) = c$$
$$g_2(b_1) = c_1$$

$$h(a) = c$$

Then $g_1 f = h = g_2 f$ though $g_1 \neq g_2$.

Exercise 1.6.v: More specifically, find a concrete category that contains a monomorphism whose underlying function is not injective. Find a concrete category that contains an epimorphism whose underlying function is not surjective.

The idea for a monic that is not injective is that there are some elements that cannot be distinguished via functions into the domain set. For example, consider a category of sets S with distinguished points $s_1, \ldots, s_n \in S$. Arrows are functions $f : S \to T$ where either $f(s_i) = t_i$ or T has one distinguished point and all $f(s_i) = t_1$ in which case the points are identified in the target set. In that second case, any function that is injective on $S \setminus \{s_i\}$ is monic in this category but not injective on the underlying sets. After all, for $g, h : Q \to S$ with $fg = fh$ then g and h must agree on $Q \setminus \{q_i\}$ by injectivity and we have $g(q_i) = s_i = h(q_i)$ by definition.

For epics that are not surjective, the idea is there is a subset that sufficiently covers the domain set so that maps out cannot disagree. The example here is the category of Hausdorff spaces with continuous maps. Any inclusion of a dense proper subset is epic but not surjective, such as the rationals in the real numbers.

Another example is $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in Ring. This is epic but not surjective. It's also monic (since it is injective) but this is not an isomorphism.

The key here is to understand that concrete categories restrict the arrows in various ways. Otherwise they would simply *be* the category of Set. When the arrows going *into* an object are restricted then it is possible to end up with monics that are not injective *out of* that object. When arrows going *out of* an object are restricted then it is possible to have epics that are not surjective into that object.

Exercise 1.6.vi: A **coalgebra** for an endofunctor $T : \mathsf{C} \to \mathsf{C}$ is an object $C \in \mathsf{C}$ equipped with a map $\gamma : C \to TC$. A morphism $f : (C, \gamma) \to (C', \gamma')$ of coalgebras is a map $f : C \to C'$ so that the square

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\gamma"'] & C' \arrow[d, "\gamma'"] \\ TC \arrow[r, "Tf"] & TC' \end{tikzcd}$$

commutes. Prove that if (C, γ) is a **terminal coalgebra**, that is a terminal object in the category of coalgebras, then the map $\gamma : C \to TC$ is an isomorphism.

Let $\gamma : C \to TC$ be a terminal coalgebra and notice that $T\gamma : TC \to TTC$ is also a coalgebra, that γ is trivially a coalgebra morphism $(C, \gamma) \to (TC, T\gamma)$, and finally since (C, γ) is terminal, we have a unique coalgebra morphism $f : (TC, T\gamma) \to (C, \gamma)$:

$$\begin{tikzcd} C \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & TC \arrow[r, "f"] \arrow[d, "T\gamma"] & C \arrow[d, "\gamma"] \\ TC \arrow[r, "T\gamma"] & TTC \arrow[r, "Tf"] & TC \end{tikzcd}$$

By the above diagram, $f\gamma : C \to C$ is a coalgebra morphism $(C, \gamma) \to (C, \gamma)$ and so must be 1_c by uniqueness since (C, γ) is terminal. Then by the right square $\gamma f = T(f\gamma) = T(1_C) = 1_{T_C}$ so that γ is an isomorphism.

Exercise 1.7.i: Prove that if C is small and D is locally small, then D^{C} is locally small by defining a monomorphism from the collection of natural transformations between a fixed pair of functors $F, G : \mathsf{C} \rightrightarrows D$ into a set.

Since C is a set, the collection $\{\alpha_c : Fc \to Gc \mid c \in \mathsf{C}\}$ is a set S_α by the axiom schema of substitution. Moreover we have that $S_\alpha \subseteq \bigcup\{\mathsf{D}(Fc, Gc) \mid c \in \mathsf{C}\}$ since each $\mathsf{D}(Fc, Gc)$ is a set.

I'll finish this one later.

Exercise 1.7.ii: Given a natural transformation $\beta : H \Rightarrow K$ and functors F and L as displayed in

$$\begin{tikzcd} C \arrow[r, "F"] & D \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & E \arrow[r, "L"] & F \end{tikzcd}$$

define a natural transformation $L\beta F : LHF \Rightarrow LKF$ by $(L\beta F)_c = L\beta_{Fc}$. This is the whiskered composited of β with L and F. Prove that $L\beta F$ is natural.

Given $f : c \to c'$, apply naturality of β to $Ff : Fc \to Fc'$ to get

$$\begin{tikzcd} HFc \arrow[r, "\beta_{Fc}"] \arrow[d, "HFf"'] & KFc \arrow[d, "KFf"] \\ HFc' \arrow[r, "\beta_{Fc'}"'] & KFc' \end{tikzcd}$$

Since functors preserve diagrams, apply L to the diagram above to get

$$\begin{tikzcd} LHFc \arrow[r, "L\beta_{Fc}"] \arrow[d, "LHFf"'] & LKFc \arrow[d, "LKFf"] \\ LHFc' \arrow[r, "L\beta_{Fc'}"'] & LKFc' \end{tikzcd}$$

which is the definition of naturality for $L\beta F$.

Exercise 1.7.iii: Redefine horizontal composition of natural transformations introduced in Lemma 1.7.4 using vertical composition and whiskering.

Given the functors and transformations

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathsf{E} \end{tikzcd}$$

define

$$\beta G \cdot H\alpha = \beta * \alpha = K\alpha \cdot \beta F$$

By the definition of whiskering, the components of the left hand side correspond to the bottom left of the following diagram and the components of the right correspond to the upper right:

$$\begin{tikzcd}
HFc \arrow[r, "\beta_{Fc}"] \arrow[d, "H\alpha_c"'] \arrow[dr, "\beta*\alpha"] & KFc \arrow[d, "K\alpha_c"] \\
HGc \arrow[r, "\beta_{Gc}"'] & KGc
\end{tikzcd}$$

which is the same definition given in LEMMA 1.7.4.

EXERCISE 1.7.iv: Prove Lemma 1.7.7: (middle four interchange) Given functors and natural transformations

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, "G"{name=G, description}] \arrow[r, bend right=50, "H"'{name=H}] & \mathsf{D} \arrow[r, bend left=50, "J"{name=J}] \arrow[r, "K"{name=K, description}] \arrow[r, bend right=50, "L"'{name=L}] & \mathsf{E}
\arrow[Rightarrow, from=F, to=G, "\alpha"]
\arrow[Rightarrow, from=G, to=H, "\beta"]
\arrow[Rightarrow, from=J, to=K, "\gamma"]
\arrow[Rightarrow, from=K, to=L, "\delta"]
\end{tikzcd}$$

the natural transformation $JF \Rightarrow LH$ defined by first composing vertically and then composing horizontally equals the natural transformation defined by first composing horizontally and then composing vertically:

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, bend right=50, "H"'{name=H}] & \mathsf{D} \arrow[r, bend left=50, "J"{name=J}] \arrow[r, bend right=50, "L"'{name=L}] & \mathsf{E}
\arrow[Rightarrow, from=F, to=H, "\beta\cdot\alpha"]
\arrow[Rightarrow, from=J, to=L, "\delta\cdot\gamma"]
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=50, "JF"{name=JF}] \arrow[r, "KG"{name=KG, description}] \arrow[r, bend right=50, "LH"'{name=LH}] & \mathsf{E}
\arrow[Rightarrow, from=JF, to=KG, "\gamma*\alpha"]
\arrow[Rightarrow, from=KG, to=LH, "\delta*\beta"]
\end{tikzcd}$$

First notice that whiskering a vertical composition with a functor distributes on the left since

$$(L(\beta\cdot\alpha))_c = L(\beta\cdot\alpha)_c = L(\beta_c\alpha_c) = L\beta_c \cdot L\alpha_c = (L\beta\cdot L\alpha)_c$$

by functoriality of L and on the right by definition:

$$((\delta\cdot\gamma)F)_c = (\delta\cdot\gamma)_{Fc} = \delta_{Fc}\cdot\gamma_{Fc} = (\delta F\cdot\gamma F)_c$$

Then,

$$\begin{aligned}
(\delta\cdot\gamma)*(\beta\cdot\alpha) &= (L(\beta\cdot\alpha))\cdot((\delta\cdot\gamma)F) && (1)\\
&= L\beta\cdot L\alpha\cdot\delta F\cdot\gamma F && (2)\\
&= L\beta\cdot(\delta*\alpha)\cdot\gamma F && (3)\\
&= L\beta\cdot\delta G\cdot K\alpha\cdot\gamma F && (4)\\
&= (\delta*\beta)\cdot(\gamma*\alpha) && (5)
\end{aligned}$$

Equation (2) is due to distributivity and the rest of the steps are applications of the previous exercise.

EXERCISE 1.7.vii: Prove that a bifunctor $F : \mathsf{C}\times\mathsf{D}\to\mathsf{E}$ determines and is uniquely determined by:

(i) A functor $F(c,-) : \mathsf{D}\to\mathsf{E}$ for each $c\in\mathsf{C}$.

(ii) A natural transformation $F(f,-) : F(c,-) \Rightarrow F(c',-)$ for each $f : c \to c'$ in $s\mathsf{C}$, defined functorially in C.

In other words, prove that there is a bijection between functors $\mathsf{C} \times \mathsf{D} \to \mathsf{E}$ and functors $\mathsf{C} \to \mathsf{E}^{\mathsf{D}}$. By symmetry of the product of categories, these classes of functors are also in bijection with functors $\mathsf{D} \to \mathsf{E}^{\mathsf{C}}$.

Assuming a bifunctor $F : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$ then define a functor $F(c,-) : \mathsf{D} \to \mathsf{E}$ by $F(c,d)$ on objects and $F(c,f) = F(1_c,f) : F(c,d) \to F(c,d')$ on arrows $f : d \to d'$. Then $F(c,gf) = F(1_c,gf) = F(1_c,g)F(1_c,f) = F(c,g)F(c,f)$ by bifunctoriality. Naturality means that for $f : c \to c'$ in C and $g : d \to d'$ in D, the following diagram commutes:

$$\begin{tikzcd}
F(c,d) \arrow[r, "F(f,d)"] \arrow[d, "F(c,g)"'] & F(c',d) \arrow[d, "F(c',g)"] \\
F(c,d') \arrow[r, "F(f,d')"'] & F(c',d')
\end{tikzcd}$$

where $F(f,d)$ is defined similarly as $F(f,1_d)$. This follows since the top-right composite is $F(1_{c'},g)F(f,1_d) = F(f,g)$ and the bottom-left composite is $F(f,1_{d'})F(1_c,g) = F(f,g)$ by bifunctoriality.

For this to be functorial in $\mathsf{C} \to \mathsf{E}^{\mathsf{D}}$ we need the composite transformation $F(gf,-) : F(c,-) \Rightarrow F(c'',-)$ to be the vertical composition $F(g,-) \cdot F(f,-)$ for $f : c \to c'$ and $g : c' \to c''$ in C. Since the definition of the vertical composite is component-wise composition, this follows since $F(gf,d) = F(g,d)F(f,d)$ by bifunctoriality.

Conversely assuming i. and ii., define a bifunctor $\mathsf{C} \times \mathsf{D} \to \mathsf{E}$ by $F(c,d)$ on objects and define $F(f,g)$ for $f : c \to c'$ in C and $g : d \to d'$ in D as the common composite, due to naturality, of the diagram

$$\begin{tikzcd}
F(c,d) \arrow[r, "F(f,d)"] \arrow[d, "F(c,g)"'] & F(c',d) \arrow[d, "F(c',g)"] \\
F(c,d') \arrow[r, "F(f,d')"'] & F(c',d')
\end{tikzcd}$$

For $h : c' \to c''$ in C and $k : d' \to d''$ in D, we show functoriality as

$$\begin{aligned}
F(hf,kg) &= F(hf,d'')F(c,kg) && \textit{By definition} \\
&= F(h,d'')F(f,d'')F(c,k)F(c,g) && \textit{By functoriality and naturality} \\
&= F(h,d'')F(c',k)F(f,d')F(c,g) && \textit{By commutativity of the definition} \\
&= F(h,k)F(f,g) && \textit{By definition}
\end{aligned}$$

Note that this exercise also solves Exercise 1.5.i by choosing $\mathsf{C} = \mathbb{2}$.

Chapter 2

Exercise 2.1.i: For each of the three functors

$$\mathbb{1} \underset{1}{\overset{0}{\underset{\longleftarrow !\,-}{\longrightarrow}}} \mathbb{2}$$

between the categories $\mathbb{1}$ and $\mathbb{2}$, describe the corresponding natural transformation between the covariant functors $\mathsf{Cat} \rightrightarrows \mathsf{Set}$ represented by the categories $\mathbb{1}$ and $\mathbb{2}$.

The functor $0 : \mathbb{1} \to \mathbb{2}$ defines the natural transformation $0^* : \mathsf{Cat}(\mathbb{2}, -) \Rightarrow \mathsf{Cat}(\mathbb{1}, -)$ by pre-composition which maps arrows to their domain objects. The functor $1 : \mathbb{1} \to \mathbb{2}$ corresponds to the natural transformation that sends arrows to their codomain objects. $! : \mathbb{2} \to \mathbb{1}$ defines the transformation that sends objects to their identity arrows.

Exercise 2.1.ii: Prove that if $F : \mathsf{C} \to \mathsf{Set}$ is representable, then F preserves monomorphisms, i.e., sends every monomorphism in C to an injective function. Use the contrapositive to find a covariant set-valued functor defined on your favorite concrete category that is not representable.

A monic $f : x \to y$ is by definition injective by post-composition $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$. Then assuming the natural isomorphism $\alpha : \mathsf{C}(c, -) \to F$, we have that $Ff = \alpha_y f_* \alpha_x^{-1}$ so that Ff is injective since it is a composition of injective functions.

$$\begin{tikzcd}
\mathsf{C}(c,x) \arrow[r, "\alpha_x"] \arrow[d, "f_*"'] & Fx \arrow[d, "Ff"] \\
\mathsf{C}(c,y) \arrow[r, "\alpha_y"] & Fy
\end{tikzcd}$$

Taking the concrete category from Exercise 1.6.v, the functor with certain identified points is not representable. That is, there is no "single point" object that we can map to the multiple points that somewhere get identified.

Exercise 2.1.iii: Suppose $F : \mathsf{C} \to \mathsf{Set}$ is equivalent to $G : \mathsf{D} \to \mathsf{Set}$ in the sense that there is an equivalence of categories $H : \mathsf{C} \to \mathsf{D}$ so that GH and F are naturally isomorphic.

(i) If G is representable, then is F representable?
(ii) If F is representable, then is G representable?

First note that whiskering preserves natural isomorphisms since the components of a whisker $(G\alpha F)_c = G\alpha_{Fc}$ which is an isomorphism since each α_d is an iso and functors preserve isos by Lemma 1.3.8. Vertical composition also preserves natural isomorphisms since the composition of the components (each isomorphisms) is an isomorphism.

Now assuming the following data

$$\begin{aligned} &K : \mathsf{D} \to \mathsf{C} \\ &\eta : 1_{\mathsf{C}} \cong KH \\ &\epsilon : HK \cong 1_{\mathsf{D}} \\ &\alpha : F \cong GH \\ &\gamma : \mathsf{C}(c, -) \cong F \end{aligned}$$

We first show that there exists $\beta : FK \cong G$ by whiskering and vertically composing the following diagram

$$\begin{tikzcd} \mathsf{D} \arrow[r, "K"] \arrow[dr, bend right, "1_{\mathsf{D}}"'] & \mathsf{C} \arrow[d, "H"] \arrow[dr, bend left, "F"] \arrow[dl, phantom, "\swArrow_{\epsilon}"] & \\ & \mathsf{D} \arrow[r, "G"'] & \mathsf{Set} \arrow[ul, phantom, "\swArrow_{\alpha}"] \end{tikzcd}$$

so that $\beta = G\epsilon \cdot \alpha K$. Thus, claims i. and ii. are equivalent. Now using β, the following diagram

$$\begin{tikzcd} \mathsf{D} \arrow[r, "K"] \arrow[rr, bend right=60, "G"'] \arrow[rr, phantom, bend right=30, "\Downarrow\beta"] & \mathsf{C} \arrow[r, bend left, "\mathsf{C}(c,-)"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathsf{Set} \end{tikzcd}$$

demonstrates that $\mathsf{C}(c, -)K \cong G$ by $\beta \cdot \gamma K$.

By Proposition 4.4.5, $\mathsf{C}(c, Kd) \cong \mathsf{D}(Hc, d)$ by the function

$$\mathsf{D}(Hc, d) \xrightarrow{K} \mathsf{C}(KHc, Kd) \xrightarrow{\eta_c^*} \mathsf{C}(c, Kd)$$

and this is natural in d. That is, $f : Hc \to d$ is mapped to $Kf \cdot \eta_c : c \to Kd$. Naturality follows because given $g : d \to d'$ the following diagram commutes

$$\begin{tikzcd} \mathsf{D}(Hc, d) \arrow[r, "K"] \arrow[d, "g_*"] & \mathsf{C}(KHc, Kd) \arrow[r, "\eta_c^*"] \arrow[d, "(Kg)_*"] & \mathsf{C}(c, Kd) \arrow[d, "(Kg)_*"] \\ \mathsf{D}(Hc, d') \arrow[r, "K"] & \mathsf{C}(KHc, Kd') \arrow[r, "\eta_c^*"] & \mathsf{C}(c, Kd') \end{tikzcd}$$

The left square commutes due to functoriality of K (the top path is $f \mapsto Kg \cdot Kf$ and bottom is $f \mapsto K(gf)$) and the right square commutes by Exercise 1.4.iv. The first map is an isomorphism because K is fully faithful and the second map is an isomorphism because η_c is. Naturality in d says that $\mathsf{D}(Hc, -) \cong \mathsf{C}(c, -)K \cong G$ so that G is represented by Hc.

Exercise 2.1.iv: Characterize subsets that assemble into subfunctors of $\mathsf{C}(-, c)$.

Let $S_x \subset \mathsf{C}(x, c)$ designate the subsets of some subfunctor of $\mathsf{C}(-, c)$. By definition, given $f \in S_c : x \to c$, for any $g : y \to x$, $g^*(f) = fg \in S_y : y \to c$. In normal language, pre-composing any arrow with a member of S_x is a member of some S_y. I suppose that characterizes these sets but here are some further observations:

(i) If $1_c \in S_c$ then $S_x = \mathsf{C}(x, c)$ for all $x \in \mathsf{C}$ because any $f : x \to c$ is the image $f^*(1_c) = 1_c f = f$ of $1_c \in S_c$.
(ii) Any iso $f \in S_c$ implies the same because then $1_c = f^{-1} f$ so that $1_c \in S_c$.
(iii) Same for any split monic with h splitting because $1_c = hf$.
(iv) Any monic $f : c' \to c$ creates a sub-functor $f_* : \mathsf{C}(-, c') \Rightarrow \mathsf{C}(-, c)$.

Exercise 2.2.i: State and prove the dual to Theorem 2.2.4, characterizing natural transformations $\mathsf{C}(-, c) \Rightarrow F$ for a contravariant functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$.

Consider the evaluation bifunctor $ev : \mathsf{C}^{\mathrm{op}} \times \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}} \to \mathsf{Set}$ defined as $ev(c, F) = Fc$ on objects and on arrows (f, α) as the common composite, due to naturality, of the following diagram

$$\begin{tikzcd}
d & Fd \arrow[r, "\alpha_d"] \arrow[d, "Ff"'] & Gd \arrow[d, "Gf"] \\
c \arrow[u, "f"] & Fc \arrow[r, "\alpha_c"'] & Gc
\end{tikzcd}$$

That is, $ev(f, \alpha) = Gf \cdot \alpha_d = \alpha_c \cdot Ff : Fd \to Gc$. In the above, we are assuming $c \in \mathsf{C}$, $F, G \in \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ which is to say that F and G are contravariant functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$, $f : c \to d$, and $\alpha : F \Rightarrow G$.

This is a functor since given $g : d \to e$ and $\beta : G \Rightarrow H$, we can compute

$$\begin{aligned}
ev(gf, \beta\alpha) &= (\beta \cdot \alpha)_c \cdot F(gf) \\
&= \beta_c \cdot \alpha_c \cdot Ff \cdot Fg \\
&= \beta_c \cdot Gf \cdot \alpha_d \cdot Fg \\
&= ev(f, \beta) \cdot ev(g, \alpha)
\end{aligned}$$

by appealing to the following diagram

$$\begin{tikzcd}
e & Fe \arrow[d, "Fg"'] \arrow[dr, dashed, "ev(g,\alpha)"] & & \\
d \arrow[u, "g"] & Fd \arrow[r, "\alpha_d"] \arrow[d, "Ff"'] & Gd \arrow[d, "Gf"'] \arrow[dr, dashed, "ev(f,\beta)"] & \\
c \arrow[u, "f"] & Fc \arrow[r, "\alpha_c"'] & Gc \arrow[r, "\beta_c"'] & Hc
\end{tikzcd}$$

and the definition above.

There is another functor $y : \mathsf{C}^{\mathrm{op}} \times \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}} \to \mathsf{Set}$ defined as $y(c, F) = \mathrm{Hom}(\mathsf{C}(-, c), F)$ on objects and on arrows as $y(f, \alpha) = \alpha_* \cdot (f_*)^* : \mathrm{Hom}(\mathsf{C}(-, d), F) \to \mathrm{Hom}(\mathsf{C}(-, c), G)$. To unpack the arrows definition, recall that any arrow $f : c \to d$ gives the natural transformation $f_* : \mathsf{C}(-, c) \Rightarrow \mathsf{C}(-, d)$ defined by post-composition: an arrow $h : b \to c$ is sent to the arrow $f_*(h) = fh : b \to d$. Thus this definition says that we pre-compose f_*

and post-compose $\alpha : F \Rightarrow G$ with a natural transformation $\delta : \mathsf{C}(-,d) \Rightarrow F$ to form $y(f,\alpha)(\delta) = \alpha \cdot \delta \cdot f_* : \mathsf{C}(-,c) \Rightarrow G$. To unpack this further, the components of this transformation are $(\alpha \cdot \delta \cdot f_*)_b(h) = \alpha_b \cdot \delta_b(fh) = \alpha_b(\delta_b(fh))$.

This too is a functor since

$$\begin{aligned} y(gf,\beta\alpha)(\delta) &= (\beta\alpha)\delta(gf)_* \\ &= \beta(\alpha\delta g_*)f_* \\ &= \beta(y(g,\alpha)(\delta))f_* \\ &= y(f,\beta)\cdot y(g,\alpha)(\delta) \end{aligned}$$

The contravariant Yoneda lemma now states that there exists a natural isomorphism $\Phi : y \Rightarrow ev$ by defining the components $\Phi_{d,F} : \mathrm{Hom}(\mathsf{C}(-,d),F) \to Fd$ by $\Phi_{d,F}(\beta) = \beta_d(1_d)$. For this to be natural, the following diagram must commute

$$\begin{tikzcd} \mathrm{Hom}(\mathsf{C}(-,d),F) \arrow[r, "\Phi_{d,F}"] \arrow[d, "y(f,\alpha)"'] & Fd \arrow[d, "ev(f,\alpha)"] \\ \mathrm{Hom}(\mathsf{C}(-,c),G) \arrow[r, "\Phi_{c,G}"'] & Gc \end{tikzcd}$$

The top right sends $\beta \mapsto \beta_d(1_d) \mapsto \alpha_c \cdot Ff(\beta_d(1_d))$ and the bottom left sends $\beta \mapsto \alpha \cdot \beta \cdot f_* \mapsto (\alpha \cdot \beta \cdot f_*)_c(1_c)$ by definition. These are the same since

$$\begin{aligned} \alpha_c \cdot Ff(\beta_d(1_d)) &= \alpha_c \cdot Ff \cdot \beta_d(1_d) \\ &= \alpha_c \cdot \beta_c \cdot f^*(1_d) && \textit{By naturality of } \beta : \mathsf{C}(-,d) \Rightarrow F \\ &= \alpha_c \cdot \beta_c(1_d f) \\ &= \alpha_c \cdot \beta_c(f 1_c) \\ &= (\alpha \cdot \beta \cdot f_*)_c(1_c) \end{aligned}$$

The inverse of $\Phi : y \Rightarrow ev$ is $\Psi : ev \Rightarrow y$ defined as follows. By definition, each $\Psi_{d,F} : Fd \to \mathrm{Hom}(\mathsf{C}(-,d),F)$ must define a natural transformation $\mathsf{C}(-,d) \Rightarrow F$ for any $x \in Fd$. Fixing d and F, we define the components of this transformation as $\Psi_{d,F}(x)_c(f) = Ff(x)$. These assemble into a natural transformation $\mathsf{C}(-,d) \to F$ by functoriality of F. That is, given $g : b \to c$

$$\begin{aligned} Fg \cdot \Psi_{d,F}(x)_c(f) &= Fg(Ff(x)) \\ &= F(fg)(x) \\ &= \Psi_{d,F}(x)_b(fg) \\ &= \Psi_{d,F}(x)_b \cdot g^*(f) \end{aligned}$$

This corresponds to the diagram:

$$\begin{tikzcd} c & \mathsf{C}(c,d) \arrow[r, "\Psi_{d,F}(x)_c"] \arrow[d, "g^*"'] & Fc \arrow[d, "Fg"] \\ b \arrow[u, "g"] & \mathsf{C}(b,d) \arrow[r, "\Psi_{d,F}(x)_b"'] & Fb \end{tikzcd}$$

Now we show that Φ and Ψ are inverses since we have both

$$\begin{aligned}
\Phi_{d,F} \cdot \Psi_{d,F}(x) &= \Psi_{d,F}(x)_d(1_d) \\
&= F1_d(x) \\
&= 1_{F_d}(x) \\
&= x
\end{aligned}$$

and

$$\begin{aligned}
(\Psi_{d,F} \cdot \Phi_{d,F}(\alpha))_c(f) &= \left(\Psi_{d,F}(\alpha_d(1_d))\right)_c(f) \\
&= Ff(\alpha_d(1_d)) \\
&= Ff \cdot \alpha_d(1_d) \\
&= \alpha_c \cdot f^*(1_d) && \textit{By naturality of } \alpha : \mathsf{C}(-,d) \Rightarrow F \\
&= \alpha_c(1_d f) \\
&= \alpha_c(f)
\end{aligned}$$

Exercise 2.2.ii: Explain why the Yoneda lemma does not dualize to classify natural transformations from an arbitrary set-valued functor to a represented functor.

Such a dualization would have to be either $\mathrm{Hom}(F, \mathsf{C}(-,c)) \cong Fc$ or $\mathrm{Hom}(F, \mathsf{C}(-,c)) \cong \mathsf{C}(c,c)$. In the first case, the variance is wrong since the codomain functor $ev : \mathsf{C}^{\mathrm{op}} \times \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}} \to \mathsf{Set}$ does not match the domain functor which would need to be

$$\begin{aligned}
y : \mathsf{C} \times \left(\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}\right)^{\mathrm{op}} &\to \mathsf{Set} \\
(c, F) &\mapsto \mathrm{Hom}(F, \mathsf{C}(-,c))
\end{aligned}$$

since the functors appear in the domain position of Hom, which is necessarily contravariant and the functor sending an object to its contravariant representation is covariant which also is wrong.

In the second case, besides the problem of not having a distinguished element in Fc to send to $\mathsf{C}(-,c)$ given a transformation $\alpha : F \to \mathsf{C}(-,c)$ there is also the problem of variance. Fixing F, ev is still contravariant in the category and y is covariant.

Exercise 2.2.iii: Prove that the contravariant Yoneda embedding $y : \omega \hookrightarrow \mathsf{Set}^{\omega^{\mathrm{op}}}$ is fully faithful.

The functor $\omega(-,n)$ is the diagram

$$\{f_{0n}\} \xleftarrow{f_{01}^*} \{f_{1n}\} \xleftarrow{f_{12}^*} \{f_{2n}\} \cdots \{f_{n-1n}\} \xleftarrow{f_{(n-1)n}^*} \{f_{nn} = 1_n\}$$

where f_{in} is the unique arrow $i \to n$.

By definition, any natural transformation $\alpha : \omega(-,n) \Rightarrow \omega(-,m)$ for $n \leq m$ must send $\alpha_i : f_{in} \mapsto f_{im}$ and is entirely determined by the image $\alpha_n(f_{nn}) = f_{nm}$ since

$$\alpha_i(f_{in}) = f_{im} = f_{nm} f_{in}$$

But $f_{nm}f_{in} = f_{nm}^*(f_{in})$ so that $\alpha = f_{nm}^*$.

If $n > m$ then $\alpha_n(f_{nn})$ must equal f_{nm} but there is no such arrow so this transformation cannot exist. Thus the natural transformations $\omega(-, n) \Rightarrow \omega(-, m)$ are in bijection with the arrows $n \to m$.

Exercise 2.2.iv: Prove the following strengthening of Lemma 1.2.3, demonstrating the equivalence between an isomorphism in a category and a **representable isomorphism** between the corresponding co- or contravariant represented functors: the following are equivalent:

(i) $f : x \to y$ is an isomorphism in C.
(ii) $f_* : \mathsf{C}(-, x) \Rightarrow \mathsf{C}(-, y)$ is a natural isomorphism.
(iii) $f^* : \mathsf{C}(y, -) \Rightarrow \mathsf{C}(x, -)$ is a natural isomorphism.

Firstly, i. $\Rightarrow$ ii. is obvious since $(f^{-1})_*$ is the inverse transformation. So assume ii. and let $\alpha : \mathsf{C}(-, y) \Rightarrow \mathsf{C}(-, x)$ be the inverse transformation. By Yoneda, α corresponds to the arrow $g = \alpha_y(1_y) : y \to x$ such that $\alpha_c(h) = h^*(g) = gh$ for $h : c \to y$. Therefore, $\alpha = g_*$. Since α is the inverse transformation to f_*, we have that $1_x = g_*f_*(1_x) = gf$ and $1_y = f_*g_*(1_y) = fg$ so that $g = f^{-1}$ and f is an isomorphism. The proof for assuming iii. is the same.

Exercise 2.2.v: By the Yoneda lemma, natural endomorphisms of the contravariant power functor $P : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ correspond bijectively to endomorphisms of its representing object $\Omega = \{\top, \bot\}$. Describe the natural endomorphisms of P that correspond to each of the four elements of $\mathrm{Hom}(\Omega, \Omega)$. Do these functions induce natural endomorphisms of the covariant power set functor?

The four endomorphisms of Ω are

(i) the identity, 1_Ω
(ii) the constant function to $\top$, $c_\top(x) = \top$.
(iii) the constant function to $\bot$, $c_\bot(x) = \bot$.
(iv) swapping $\top$ and $\bot$, $\mathrm{swap}(\top) = \bot$, $\mathrm{swap}(\bot) = \top$.

and they each induce natural transformations of $\mathsf{Set}(-, \Omega)$ by post-composition. Given $f : S \to \Omega$ corresponding to the subset $f^{-1}(\top) \subset S$, post-composition with (ii), sends everything to $\top$ which corresponds to the subset $S \subseteq S$. Thus this transformation $c_{\top *}$ corresponds to sending every subset of S to S. Similarly, iii sends every subset of S to the empty set $\{\}$. The endomorphism swap sends every subset $U \subseteq S$ to its complement $S \setminus U$.

Considering the covariant power set functor, sending every subset $U \subseteq S$ to S is not natural since given $f : S \to T$, we need that $U \mapsto S \mapsto f(S)$ is equal to $f(U) \mapsto T$ which is only true if f is surjective, i.e., $f(S) = T$. Sending every subset to the empty set is natural since $U \mapsto \{\} \mapsto f(\{\}) = \{\}$. Finally, sending every subset to its complement is not natural since we would need $f(S \setminus U) = T \setminus f(U)$. If $T = \{x, y\}$ and f the constant function to x, then $f(S \setminus U)$ will still be $\{x\}$ as opposed to $T \setminus f(U) = \{y\}$.

Exercise 2.2.vi: Do there exist any non-identity natural endomorphisms of the category of spaces? That is, does there exist any family of continuous maps $X \to X$, defined for all spaces X and not all of which are identities, that are natural in all maps in the category Top?

Any such transformation would compose, i.e. via whiskering, with the forgetful functor $U : \mathsf{Top} \to \mathsf{Set}$ to give a non-identity natural transformation $U \Rightarrow U$. By Yoneda, this must correspond to a non-identity endomorphism of the representing object, the singleton space with one element. Such an endomorphism cannot exist.

Exercise 2.2.vii: Use the Yoneda lemma to explain the connection between homeomorphisms of the standard unit interval $I = [0,1] \subset \mathbb{R}$ and natural automorphisms of the path functor Path : $\mathsf{Top} \to \mathsf{Set}$ of Example 2.1.5(xiv), which we call **re-parametrizations.**

Any homeomorphism $f : I \to I$ defines the natural endomorphism of Path via $f^*(p) = pf$ for $p : I \to X$. Recalling that each α_X is a bijective function from $\mathsf{Top}(I,X) \to \mathsf{Top}(I,X)$, any natural endomorphism α : Path $\Rightarrow$ Path defines the morphism $\alpha_I(1_I) : I \to I$ which is an automorphism by Exercise 2.2.iv.

Exercise 2.3.i: What are the universal elements for the representations:

(i) defining the category $\mathbb{2}$ in Examples 2.1.5(x)?
(ii) defining the Sierpinski space in Example 2.1.6(ii)?
(iii) defining the Sierpinski space in Example 2.1.6(iii)?

Solution:

(i) For the representation $\mathsf{Cat}(\mathbb{2}, -) \Rightarrow \mathrm{mor}$, the universal element is the single morphism $0 \to 1 \in \mathrm{mor}(\mathbb{2})$.

Exercise 2.3.iii: The set B^A of functions from a set A to a set B represents the contravariant functor $\mathsf{Set}(- \times A, B) : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$. The universal element for this representation is a function

$$\mathrm{ev} : B^A \times A \to B$$

called the **evaluation map**. Define the evaluation map and describe its universal property, in analogy with the universal bilinear map $\otimes$ of Example 2.3.7.

Solution: The natural isomorphism $\mathsf{Set}(-, B^A) \Rightarrow \mathsf{Set}(- \times A, B)$, sends a function $f : C \to B^A$ to the dyadic function $(c,a) \mapsto f(c)(a) \in B$ for $c \in C$ and $a \in A$. Thus, the identity 1_{B^A} is sent to the universal element function $\mathrm{ev} : B^A \times A \to B$ which, explicitly, is

$$\mathrm{ev}(g,a) = 1_{B^A}(g)(a) = g(a)$$

for any $g : A \to B$ and $a \in A$.

We say in this case that $\mathrm{ev} : B^A \times A \to B$ is the universal function in two variables whose second variable are elements of A and whose domain is B. It is the universal element in the set of all such functions in the sense that any $f : C \times A \to B$ is uniquely determined by a function $\bar{f} : C \to B^A$ such that the following diagram commutes:

$$\begin{tikzcd} C \times A \arrow[r, "\bar{f}\times 1_A"] \arrow[dr, "f"'] & B^A \times A \arrow[d, "\mathrm{ev}"] \\ & B \end{tikzcd}$$

That is, $f(c,a) = \mathrm{ev}(\bar{f}(c), a) = \bar{f}(c)(a)$ which serves as a definition for $\bar{f}$ by defining $\bar{f}(c)$ for every $c \in \mathsf{C}$.

Exercise 2.4.i: Given $F : \mathsf{C} \to \mathsf{Set}$, show that $\int F$ is isomorphic to the comma category $* \downarrow F$ of the singleton set $* : \mathbb{1} \to \mathsf{Set}$ over the functor $F : \mathsf{C} \to \mathsf{Set}$.

Solution: Objects of $* \downarrow F$ are tuples $(*, c, x)$ where $*$ is the singleton set, $c \in \mathsf{C}$ and $x : * \to Fc$ which is to say that x selects a single element $x \in Fc$. Thus the objects are the same as those of $\int F$.

Since 1_* is the only arrow $* \to *$, an arrow $(*, c, x) \to (*, c', x')$ in $* \downarrow F$ is a tuple $(1_*, f : c \to c')$ such that the following diagram commutes

$$\begin{tikzcd} * \arrow[r, "x"] \arrow[dr, "x'"'] & Fc \arrow[d, "Ff"] & c \arrow[d, "f"] \\ & Fc' & c' \end{tikzcd}$$

Since x and x' are seen as selecting elements $x \in Fc$ and $x' \in Fc'$, we have that $Ff(x) = x'$. Thus the arrows are the same as in $\int F$.

Contravariantly, for $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$, we have that the objects of $* \downarrow F$ are the same objects as before and arrows are tuples $(1_*, f : c \to c')$ such that the following diagram commutes

$$\begin{tikzcd} * \arrow[r, "x"] \arrow[dr, "x'"'] & Fc & c \arrow[d, "f"] \\ & Fc' \arrow[u, "Ff"'] & c' \end{tikzcd}$$

Thus we have that $\int F = * \downarrow F$ in the contravariant case as well.

Exercise 2.4.ii: Characterize the terminal objects of C/c.

Solution: A terminal object is an arrow $f_t : t \to c$ such that any arrow f to c uniquely factors through f_t:

$$\begin{tikzcd} x \arrow[d, "\exists! \bar{f}"'] \arrow[dr, "f"] & \\ t \arrow[r, "f_t"'] & c \end{tikzcd}$$

That is, $f = f_t \bar{f}$. The arrow 1_c is such a terminal object since we always have

$$\begin{tikzcd} x \arrow[d, "f"'] \arrow[dr, "f"] & \\ c \arrow[r, "1_c"'] & c \end{tikzcd}$$

so that in this case $\bar{f} = f$.

We claim that the domain object t of any such terminal arrow f_t is isomorphic to c. Since 1_c and f_t are both terminal, they are uniquely isomorphic by Exercise 1.6.ii. Because composition in C/c is just composition in C, isomorphisms in C/c are commuting isomorphisms in C. Thus we must have an isomorphism $\iota : t \to c$ such that the following diagram commutes:

$$\begin{tikzcd} t \arrow[d, "\iota"'] \arrow[dr, "f_t"] & \\ c \arrow[r, "1_c"] & c \end{tikzcd}$$

Thus $f_t : t \to c$ is that unique isomorphism.