Chapter 1

Exercise 1.2: A C^∞ function very flat at 0.

Let $f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$

(i) Show by induction that for $x > 0$ and $k \geq 0$, the kth derivative $f^{(k)}(x)$ is of the form $p_{2k}(1/x)e^{-1/x}$ for some polynomial $p_{2k}(y)$ of degree $2k$ in y.

(ii) Prove that f is C^∞ on $\mathbb{R}$ and that $f^{(k)}(0) = 0$ for all $k \geq 0$.

Solution:

(i) The base case is trivial so assume $f^{(k)}(x) = p_{2k}\left(\frac{1}{x}\right)e^{-1/x}$. Then

$$\begin{aligned} f^{(k+1)} &= -\frac{1}{x^2}p'_{2k}\left(\frac{1}{x}\right)e^{-1/x} + p_{2k}\left(\frac{1}{x}\right)\cdot\left(\frac{1}{x^2}\right)e^{-1/x} \\ &= \left(\frac{1}{x^2}\right)\left(-p'_{2k}\left(\frac{1}{x}\right) + p_{2k}\left(\frac{1}{x}\right)\right)\cdot e^{-1/x} \end{aligned}$$

Since $p'_{2k}(y)$ is of degree $2k-1$ it cannot cancel the highest degree term in $p_{2k}(y)$ so that the sum is still of degree $2k$. The term $1/x^2$ corresponds to the polynomial y^2 of degree 2 so that multiplication with $-p'_{2k}(1/x) + p_{2k}(1/x)$ is a polynomial in $1/x$ of degree $2k+2$.

(ii) Given a polynomial $p_n(y) = \sum_{i=0}^n c_i y^i$ we have

$$p_n(y) \leq cy^n$$

for $y \geq 1$ where $c = \max\{c_i\}\cdot(n+1)$. Given a large enough y we can have that

$$cy^n \leq \varepsilon\frac{1}{(n+1)!}y^{n+1}$$

for any ε. Specifically, choose any $y \geq \frac{c(n+1)!}{\varepsilon}$. Then

$$\varepsilon\frac{1}{(n+1)!}y^{n+1} < \varepsilon\sum_{n=0}^{\infty}\frac{1}{n!}y^n = \varepsilon e^y$$

also for $y \geq 1$. Combining these inequalities, we have that

$$p_n(y)e^{-y} < \varepsilon$$

as long as $y \geq \max\left\{1, \frac{c(n+1)!}{\varepsilon}\right\}$. Since ε may be arbitrarily small, we have that

$$\lim_{y\to\infty} p_n(y)e^{-y} = 0$$

By choosing $n = 2k$ and noting that

$$\lim_{x\to 0}\left(y = \frac{1}{x}\right) = \infty$$

we have that $f^{(k)}(0) = 0$ for all k.

Exercise 1.3: A diffeomorphism of an open interval with $\mathbb{R}$.

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ be open subsets. A C^∞ map $F : U \to V$ is called a *diffeomorphism* if it is bijective and has a C^∞ inverse $F^{-1} : V \to U$.

(i) Show that the function $f :]-\pi/2, \pi/2[\to \mathbb{R}, f(x) = \tan x$ is a diffeomorphism.

(ii) Let a, b be real numbers with $a < b$. Find a linear function $h :]a, b[\to]-1, 1[$, thus proving that any two finite open intervals are diffeomorphic.

Solution:

(i) Given that
- $\lim_{x \to \frac{\pi}{2}} \cos x = 0$
- $0 < \sin c < \sin x$ for $0 < c < x < \frac{\pi}{2}$
- $0 < \cos x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

we have that $0 < \frac{\sin c}{\cos x} < \frac{\sin x}{\cos x}$ for $0 < c < x < \pi/2$, choosing some small non-zero c. Thus:

$$\infty = \lim_{x \to \frac{\pi}{2}} \frac{\sin c}{\cos x} < \lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\cos x} = \lim_{x \to \frac{\pi}{2}} \tan x$$

Similarly, $\lim_{x \to -\frac{\pi}{2}} \tan x = -\infty$. Since $\tan x$ is differentiable, it is continuous so that it assumes all values in $\mathbb{R}$ by the intermediate value theorem.

The derivative of tan is

$$\begin{aligned}\tan' x &= \frac{\sin' x \cos x - \sin x \cos' x}{(\cos x)^2} \\ &= \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2} \\ &= 1 + \left(\frac{\sin x}{\cos x}\right)^2 \\ &= 1 + (\tan x)^2\end{aligned}$$

This is smooth since successive derivatives will be polynomials in $\tan x$. That is, assuming $\tan^{(k)} x = p_{k+1}(\tan x)$ where $p_{k+1}(x)$ is a polynomial of degree $k+1$ in x, we have that

$$\begin{aligned}\tan^{(k+1)} x &= \left(1 + (\tan x)^2\right) p'_{k+1}(\tan x) \\ &= p'_{k+1}(\tan x) + p'_{k+1}(\tan x) \cdot (\tan x)^2\end{aligned}$$

which is a polynomial in $\tan x$ of degree $k+2$. Thus all the derivatives exist and are defined on $]-\pi/2, \pi/2[$.

Since $y^2 \geq 0$ for any $y \in \mathbb{R}$, we have that $1 + (\tan x)^2 \geq 1$. By the inverse function theorem, we have that the inverse $\tan^{-1} = \arctan$ exists for any neighborhood $U \subset \mathbb{R}$ of any point $y = \tan x$ for some $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

which is all of $\mathbb{R}$ by the previous discussion. Since $x = \tan(\arctan x)$, we can calculate the derivative as follows:

$$\begin{aligned} 1 &= \tan'(\arctan x) \cdot \arctan' x \\ &= \left(1 + (\tan(\arctan x))^2\right) \cdot \arctan' x \\ &= \left(1 + x^2\right) \cdot \arctan' x \end{aligned}$$

So that $\arctan' x = \frac{1}{1+x^2}$. By the fundamental theorem of calculus, we have that

$$\arctan x = \int_0^x \frac{1}{1+t^2} dt$$

given that $\arctan 0 = \arctan(\tan 0) = 0$. This is smooth because we can show that all the derivatives $\arctan^{(k)} x = \frac{p(x)}{(1+x^2)^k}$ where $p(x)$ is a polynomial in x. Verifying this inductively:

$$\begin{aligned} \arctan^{(k+1)} &= \frac{p'(x) \cdot \left(1 + x^2\right)^k - p(x) \cdot k\left(1 + x^2\right)^{k-1} \cdot 2x}{(1+x^2)^{2k}} \\ &= \frac{\left(p'(x)\left(1 + x^2\right) - 2kxp(x)\right)\left(1 + x^2\right)^{k-1}}{(1+x^2)^{2k}} \\ &= \frac{p'(x)\left(1 + x^2\right) - 2kxp(x)}{(1+x^2)^{k+1}} \end{aligned}$$

Thus, since $1 + x^2$ is never zero, all the derivatives $\arctan^{(k+1)}$ are defined and differentiable for all of $\mathbb{R}$. Thus tan is a diffeomorphism $\left]-\frac{\pi}{2}, -\frac{\pi}{2}\right[\xrightarrow{\cong} \mathbb{R}$.

(ii) Let $h(x) = \frac{2x-b-a}{b-a}$. Then

$$h(a) = \frac{2a-b-a}{b-a} = \frac{-(b-a)}{b-a} = -1$$

and

$$h(b) = \frac{2b-b-a}{b-a} = \frac{b-a}{b-a} = 1$$

.

A Digression on Derivations and the Yoneda Lemma

A fundamental theorem of differential geometry is that the vector space of derivations at a point is homeomorphic $\mathbb{R}^n$. Somewhat surprisingly, this homeomorphism is a consequence of the Yoneda lemma, demonstrating the breadth of this lemma's application. In order to apply the Yoneda lemma all we must do is find the correct set-valued functor, which in this case is the derivative functor taking a smooth map to its Jacobian.

The crucial step of the proof is to show that a derivation at a point is a natural transformation which follows by applying the same fact, Taylor's theorem with remainder, which is used to establish the typical bijection between derivations and vectors.

Definition 1.1. Let $C_p^\infty(\mathbb{R}^n)$ be the set of germs of smooth functions on $\mathbb{R}^n$ at p with codomain $\mathbb{R}$. A **derivation at p** is a linear map $D : C_p^\infty(\mathbb{R}^n) \to \mathbb{R}$ such that

$$D(f \cdot g) = Df \cdot g(p) + f(p) \cdot Dg$$

Let C_*^∞ be the category with objects the euclidean spaces $\mathbb{R}^n$ with a chosen basepoint $p \in \mathbb{R}^n$, denoted $\mathbb{R}_p^n$, and with arrows $\mathbb{R}_p^n \to \mathbb{R}_q^m$ germs of smooth functions $f : U \to \mathbb{R}_q^m$ at $p \in U$ such that $q = f(p)$.

Theorem 1.2. *The set of derivations $\mathbf{D}(\mathbb{R}_p^n)$, define a functor $C_*^\infty \to \mathsf{Vect}$.*

Proof. Define addition and scalar multiplication on $\mathbf{D}(\mathbb{R}_p^n)$ pointwise. The proof that derivations form a vector space can be found in Loring Tu, *An Introduction to Manifolds* (techinically an exercise) so we only prove functoriality as follows.

Given $f \in C_*^\infty(\mathbb{R}_p^n, \mathbb{R}_q^m)$ define $\mathbf{D}(f) : \mathbf{D}(\mathbb{R}_p^n) \to \mathbf{D}(\mathbb{R}_q^m)$ by $f^*(D)(g) = D(gf)$ for $D \in \mathbf{D}(\mathbb{R}_p^n)$ and $g \in C_q^\infty(\mathbb{R}^m)$. This is a derivation $\mathbf{D}(\mathbb{R}_q^m)$ since

$$\begin{aligned} f^*(D)(g \cdot h) &= D((g \cdot h)f) \\ &= D(gf \cdot hf) \\ &= g(f(p)) \cdot D(hf) + D(gf) \cdot h(f(p)) \\ &= g(q) \cdot f^*(D)h + f^*(D)g \cdot h(q) \end{aligned}$$

□

Let $T_p(\mathbb{R}^n) \in \mathsf{Vect}$ be the tangent space at p. That is, $T_p(\mathbb{R}^n) = (\mathbb{R}^n, p)$, the vector space $\mathbb{R}^n$ coupled to a distinguished point $p \in \mathbb{R}^n$. Define the functor $T : C_*^\infty \to \mathsf{Vect}$ by $T(\mathbb{R}_p^n) = T_p\mathbb{R}^n$ on objects and on arrows by $T(f) = \Delta_p f$, the total derivative of f at p. Now, given any $v \in T_p\mathbb{R}^n$s we can define a linear map $\delta_{\mathbb{R}_p^n}(v) : C_p^\infty(\mathbb{R}^n) \to \mathbb{R}$ by $\delta_{\mathbb{R}_p^n}(v)f = \Delta_p f \cdot v$. In fact we have that:

Theorem 1.3. *The map $\delta_{\mathbb{R}_p^n}(v) : C_p^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation and the functions $\delta_{\mathbb{R}_p^n} : T_p\mathbb{R}^n \to \mathbf{D}(\mathbb{R}_p^n)$ define a natural transformation $\delta : T \Rightarrow \mathbf{D}$.*

Proof. The map $\delta_{\mathbb{R}_p^n}(v)$ is a derivation at p because

$$\begin{aligned} \delta_{\mathbb{R}_p^n}(v)(f \cdot g) &= \Delta_p(f \cdot g) \cdot v \\ &= \big(g(p) \cdot \Delta_p f + f(p) \cdot \Delta_p g\big) \cdot v \\ &= g(p) \cdot \Delta_p f \cdot v + f(p) \cdot \Delta_p g \cdot v \\ &= g(p) \cdot \delta_{\mathbb{R}_p^n}(v)f + f(p) \cdot \delta_{\mathbb{R}_p^n}(v)g \\ &= \delta_{\mathbb{R}_p^n}(v)f \cdot g(p) + f(p) \cdot \delta_{\mathbb{R}_p^n}(v)g \end{aligned}$$

by the Leibniz product rule for total derivatives.

For naturality, we must show that, for $f \in C_*^\infty(\mathbb{R}_p^n, \mathbb{R}_q^m)$, the following diagram commutes

$$\begin{tikzcd} T_p\mathbb{R}^n \arrow[r, "\delta_{\mathbb{R}_p^n}"] \arrow[d, "\Delta_p f"'] & \mathbf{D}\left(\mathbb{R}_p^n\right) \arrow[d, "\mathbf{D}(f)"] \\ T_q\mathbb{R}^m \arrow[r, "\delta_{\mathbb{R}_q^m}"'] & \mathbf{D}\left(\mathbb{R}_q^m\right) \end{tikzcd}$$

Thus given any $v \in T_p\mathbb{R}^n$ and $g \in C_q^\infty(\mathbb{R}^m)$ we can compute, using the chain rule for total derivatives, that

$$\begin{aligned} \delta_{\mathbb{R}_q^m}(\Delta_p f \cdot v)g &= \Delta_q g \cdot \Delta_p f \cdot v \\ &= \Delta_p(gf) \cdot v \\ &= \delta_{\mathbb{R}_p^n}(v)(gf) \\ &= f^*\left(\delta_{\mathbb{R}_p^n}(v)\right)g \\ &= \mathbf{D}(f)\left(\delta_{\mathbb{R}_p^n}(v)\right)g \end{aligned}$$

so that $\delta_{\mathbb{R}_q^m} \circ \Delta_p f = \mathbf{D}(f) \circ \delta_{\mathbb{R}_p^n}$ □

Theorem 1.4. *Derivations at a point are naturally isomorphic to natural transformations $\mathfrak{D} : C_*^\infty(\mathbb{R}_p^n, -) \Rightarrow T$.*

Proof. First we show that any natural transformation $\mathfrak{D} : C_*^\infty(\mathbb{R}_p^n, -) \Rightarrow T$ defines a derivation D at p by $Df = \mathfrak{D}_{\mathbb{R}_{f(p)}}(f)$ for $f \in C_p^\infty(\mathbb{R}^n)$. For the following proofs we will repeatedly refer to the following diagram

$$\begin{tikzcd} C_*^\infty\left(\mathbb{R}_p^n, \mathbb{R}_p^n\right) \arrow[r, "\mathfrak{D}_{\mathbb{R}_p^n}"] \arrow[d, "f_*"'] & T_p\mathbb{R}^n \arrow[d, "\Delta_p f"] \\ C_*^\infty\left(\mathbb{R}_p^n, \mathbb{R}_q\right) \arrow[r, "\mathfrak{D}_{\mathbb{R}_q}"'] & T_q\mathbb{R} \end{tikzcd} \tag{1}$$

and in particular the result of tracing the identity $1_{\mathbb{R}_p^n}(x) = x$ around the commutative square to give $\mathfrak{D}_{\mathbb{R}_q} f = \Delta_p f \cdot \mathfrak{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n}$ for various values of $f : C_p^\infty(\mathbb{R}^n)$. First, taking $f(x) = g(x) + h(x)$ for two functions $g \in C_*^\infty\left(\mathbb{R}_p^n, \mathbb{R}_{g(p)}\right)$ and $h \in C_*^\infty\left(\mathbb{R}_p^n, \mathbb{R}_{h(p)}\right)$, we have that

$$\begin{aligned} \mathfrak{D}_{\mathbb{R}_q}(g + h) &= \Delta_p(g + h) \cdot \mathfrak{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n} \\ &= \left(\Delta_p g + \Delta_p h\right) \cdot \mathfrak{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n} \\ &= \Delta_p g \cdot \mathfrak{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n} + \Delta_p h \cdot \mathfrak{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n} \\ &= \mathfrak{D}_{\mathbb{R}_{g(p)}} g + \mathfrak{D}_{\mathbb{R}_{h(p)}} h \end{aligned}$$

by linearity of the derivative. Thus D is linear.

Now, by taking $f(x) = g(x) \cdot f(x)$, we have

$$\begin{aligned}
\mathfrak{D}_{\mathbb{R}_q}(g \cdot h) &= \Delta_p(g \cdot h) \cdot \mathfrak{D}_{\mathbb{R}^n_p} 1_{\mathbb{R}^n_p} \\
&= \big(g(p)\Delta_p h + h(p)\Delta_p g\big) \cdot \mathfrak{D}_{\mathbb{R}^n_p} 1_{\mathbb{R}^n_p} \\
&= g(p)\Delta_p h \cdot \mathfrak{D}_{\mathbb{R}^n_p} 1_{\mathbb{R}^n_p} + h(p)\Delta_p g \cdot \mathfrak{D}_{\mathbb{R}^n_p} 1_{\mathbb{R}^n_p} \\
&= g(p)\mathfrak{D}_{\mathbb{R}_{h(p)}} h + h(p)\mathfrak{D}_{\mathbb{R}_{g(p)}} g
\end{aligned}$$

by the Leibniz product rule for the derivative. Thus, $Df = \mathfrak{D}_{\mathbb{R}_{f(p)}}(f)$ is a point-derivation at p.

Conversely, we show that a point-derivation D at p defines a natural transformation $\mathfrak{D} : C^\infty_*(\mathbb{R}^n_p, -) \Rightarrow T$ as follows. Given $f \in C^\infty_*\left(\mathbb{R}^n_p, \mathbb{R}^m_q\right)$ we can write $f(x) = \sum f^j(x)e_j$ for m germs of functions $f^j \in C^\infty_p(R^n)$, given a chosen basis e_j for $\mathbb{R}^n$. Define $\mathfrak{D}_{\mathbb{R}^m_q} f = \sum Df^j e_j \in T_q\mathbb{R}^m$ which we will also write as Df or $D_x f(x)$. Note that this is linear and zero on constant functions since the same is true for D on each component (see Tu, *An Introduction to Manifolds*).

By Taylor's theorem with remainder, we can write f as

$$f(x) = f(p) + \sum \left(x^i - p^i\right) g_i(x)$$

where $x = \sum x^i e_i$, $p = \sum p^i e_i$, each $g_i \in C^\infty_*\left(\mathbb{R}^n_p, \mathbb{R}^m_{g_i(p)}\right)$, and $g_i(p) = \frac{\partial f}{\partial x^i}(p)$. Then we have that

$$\begin{aligned}
\mathfrak{D}_{\mathbb{R}^m_q} f &= D_x f(p) + \sum D_x\left(\left(x^i - p^i\right) g_i(x)\right) \\
&= 0 + \sum (p^i - p^i) \cdot D_x g_i(x) + Dx^i \cdot g_i(p) \\
&= \sum Dx^i \cdot \frac{\partial f}{\partial x^i}(p) \\
&= \Delta_p f \cdot D_x x \\
&= \Delta_p f \cdot \mathfrak{D}_{\mathbb{R}^n_p} 1_{\mathbb{R}^n_p} \\
&= T(f) \cdot \mathfrak{D}_{\mathbb{R}^n_p} 1_{\mathbb{R}^n_p}
\end{aligned}$$

This shows that each $\mathfrak{D}_{\mathbb{R}^m_q}$ is the same as the corresponding component of the natural transformation defined, due to the Yoneda lemma, by the element $\mathfrak{D}_{\mathbb{R}^n_p} 1_{\mathbb{R}^n_p}$ so that $\mathfrak{D}$ is natural.

Furthermore, this correspondence is a bijection since given $\mathfrak{D} : C^\infty_*(\mathbb{R}^n_p, -) \Rightarrow T$ and defining $Df = \mathfrak{D}_{\mathbb{R}_q} f$, we have that $Dx^i = \Delta_p x^i \mathfrak{D}_{\mathbb{R}^n_p} 1_{\mathbb{R}^n_p}$ by taking $f(x) = x^i$ in diagram (1). Choosing α^j to be the dual basis to e_j we can write the linear transformation $\Delta_p x^i$ as $\sum_j \frac{\partial x^i}{\partial x^j}\alpha^j$. But $\frac{\partial x^i}{\partial x^j}$ is the Kronecker delta δ^i_j so that $\Delta_p x^i = \alpha^i$. Using this fact to recover the natural transformation, we have, by definition of D_x, that

$$\begin{aligned} D_x x &= \sum_i D x^i e_i \\ &= \sum_i \left(\Delta_p x^i \cdot \mathfrak{D}_{\mathbb{R}^n_p} 1_{\mathbb{R}^n_p}\right) e_i \\ &= \sum_i \alpha^i \left(\mathfrak{D}_{\mathbb{R}^n_p} 1_{\mathbb{R}^n_p}\right) e_i \\ &= \mathfrak{D}_{\mathbb{R}^n_p} 1_{\mathbb{R}^n_p} \end{aligned}$$

Thus D defines the natural transformation determined by the same tangent vector $\mathfrak{D}_{\mathbb{R}^n_p} 1_{\mathbb{R}^n_p}$ that, by Yoneda, must determine $\mathfrak{D}$. The other direction is trivial since we can recover D as simply the one-dimensional component of the natural transformation $\mathfrak{D}$ it defines. □

By applying the Yoneda lemma to the previous theorem, we have:

Corollary 1.5. *The natural transformation $\delta : T \Rightarrow \mathbf{D}$ defined in Theorem 1.3 is an isomorphism.*

Proof. The Yoneda lemma gives us a natural isomorphism

$$\Psi : T_p\mathbb{R}^n \to \mathrm{Hom}(C^\infty_*(\mathbb{R}^n_p, -), T)$$

defined by $\Psi(x)_{\mathbb{R}^m_q}(f) = \Delta_p f \cdot x$ for $x \in T_p\mathbb{R}^n$ and $f \in C^\infty_*\left(\mathbb{R}^n_p, \mathbb{R}^m_q\right)$. The previous theorem shows that the one-dimensional components of this transformation $\Psi(x)$ establish a bijection with derivations at p which are exactly the maps $\delta_{\mathbb{R}^n_p}$ from Theorem 1.3. □

Exercise 2.1: Vector fields

Let X be the vector field $x\partial/\partial x + y\partial/\partial y$. and $f(x, y, z) = x^2 + y^2 + z^2$. Compute Xf.

$$Xf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = x \cdot 2x + y \cdot 2y = 2x^2 + 2y^2$$

Exercise 2.4: Product of derivations

Let A be an algebra over a field K. If D_1 and D_2 are derivations of A, show that $D_1 \cdot D_2$ is not necessarily a derivation (it is if D_1 or $D_2 = 0$), but $D_1 \cdot D_2 - D_2 \cdot D_1$ is always a derivation of A.

Take $f(x, y) = (x+1)(y+1) = xy - x + y + 1$ and let $D_1 = \left.\frac{\partial}{\partial x}\right|_0$ and $D_2 = \left.\frac{\partial}{\partial y}\right|_0$. Then, given that $\frac{\partial f}{\partial x} = y + 1$ and $\frac{\partial f}{\partial y} = x + 1$, we have

$$(D_1 \cdot D_2) f = \left.\frac{\partial f}{\partial x}\right|_0 \cdot \left.\frac{\partial f}{\partial y}\right|_0 = 1 \cdot 1 = 1$$

but this must equal

$$
\begin{aligned}
(D_1 \cdot D_2)f &= (x+1)\cdot\big|_0 \left.\frac{\partial(y+1)}{\partial x}\right|_0 \cdot \left.\frac{\partial(y+1)}{\partial y}\right|_0 + \left.\frac{\partial(x+1)}{\partial x}\right|_0 \cdot \left.\frac{\partial(x+1)}{\partial y}\right|_0 \cdot(y+1)\big|_0 \\
&= 1\cdot 0\cdot 1 + 1\cdot 0\cdot 1 \\
&= 0+0=0
\end{aligned}
$$

so that $D_1 \cdot D_2$ is not a derivation.

Exercise 3.1: Tensor product of covectors Let $e_1, \dots, e_n$ be a basis for a vector space V and let $\alpha^1, \dots, \alpha^n$ be its dual basis in V^*. Suppose $[g_{ij}] \in \mathbb{R}^{n\times n}$ is an $n \times n$ matrix. Define a bilinear function $f : V \times V \to \mathbb{R}$ by

$$f(v,w) = \sum_{1\le i,j\le n} g_{ij} v^i w^j$$

for $v = \sum v^i e_i$ and $w = \sum w^j e_j$ in V. Describe f in terms of the tensor products of α^i and α^j, $1 \le i, j \le n$.

Solution:

$$f = \sum_{1\le i,j\le n} g_{ij} \alpha^i \otimes \alpha^j$$

Further, if we define

$$h = \sum_{1\le i,j\le n} g_{ij} \alpha^i \wedge \alpha^j$$

then we have that

$$
\begin{aligned}
h(v,w) &= \sum_{1\le i,j\le n} g_{ij} \left(v^i w^j - w^i v^j\right) \\
&= \sum_{1\le i,j\le n} g_{ij} v^i w^j - \sum_{1\le i,j\le n} g_{ij} v^j w^i \\
&= \sum_{1\le i,j\le n} g_{ij} v^i w^j - \sum_{1\le i,j\le n} g_{ji} v^i w^j \\
&= \sum_{1\le i,j\le n} \left(g_{ij} - g_{ji}\right) v^i w^j
\end{aligned}
$$

so that $h = \sum \left(g_{ij} - g_{ji}\right) \alpha^i \otimes \alpha^j$ is determined by the matrix $[h_{ij}] = [g_{ij} - g_{ji}]$ which has the properties $h_{ii} = 0$ and $h_{ij} = -h_{ji}$. Thus, such 2-covectors are determined by the upper-right values excluding the diagonal and there are $\binom{n}{2}$ choices for such values as expected.

Exercise 3.2: Hyperplanes

(i) Let V be a vector space of dimension n and $f : V \to \mathbb{R}$ a nonzero linear functional. Show that $\dim \ker f = n - 1$.

(ii) Show that a nonzero linear functional on a vector space V is determined up to a multiplicative constant by its kernel, a hyperplane in V. In other words, if f and $g : V \to \mathbb{R}$ are nonzero linear functionals and $\ker f = \ker g$, then $g = cf$ for some constant $c \in \mathbb{R}$.

Solution:

(i) Let $e_1, \dots, e_n$ be a basis for V and $\alpha^1, \dots, \alpha^n$ the dual basis so that $f = \sum c_i \alpha^i$ for some $c_i \in \mathbb{R}$. Since f is not zero, not all of the c_i can be zero so assume, without loss of generality, that $c_n \neq 0$. Then, given any $v = \sum v^i e_i \in \ker f$, we have $\sum c_i v^i = 0$ so that

$$v^n = -\sum_{i=1}^{n-1} \frac{c_i v^i}{c_n}$$

This shows that the $n-1$ vectors $f_i = e_i - \frac{c_i}{c_n} e_n$ for $1 \leq i \leq n-1$ span $\ker f$ since v is exactly the following sum

$$\begin{aligned}
\sum_{i=1}^{n-1} v^i f_i &= \sum_{i=1}^{n-1} \left(v^i e_i - \frac{c_i v^i}{c_n} e_n \right) \\
&= \sum_{i=1}^{n-1} v^i e_i - \sum_{i=1}^{n-1} \frac{c_i v^i}{c_n} e_n \\
&= \sum_{i=1}^{n-1} v^i e_i + v^n e_n \\
&= v
\end{aligned}$$

For linear independence, suppose that $w = \sum w^i f_i = 0$. Then

$$0 = \alpha^j(w) = \sum_{i=1}^{n-1} \left(w^j \alpha^j(e_i) - \frac{c_i w^i}{c_n} \alpha^j(e_n) \right) = \sum_{i=1}^{n-1} \left(w^j \delta_i^j - \frac{c_i w^i}{c_n} \delta_n^j \right) = w^j$$

for $1 \leq j \leq n-1$ so that the f_i are independent.

(ii) For any $v \in V$ we have that

$$\begin{aligned}
v &= \sum_{i=1}^{n} v^i e_i \\
&= \sum_{i=1}^{n-1} \left(v^i e_i - \frac{c_i v^i}{c_n} e_n \right) + \sum_{i=1}^{n-1} \frac{c_i v^i}{c_n} e_n + v^n e_n \\
&= \sum_{i=1}^{n-1} v^i f_i + \sum_{i=1}^{n-1} \frac{c_i v^i}{c_n} e_n + \frac{c_n v^n}{c_n} e_n \\
&= v_0 + \sum_{i=1}^{n} \frac{c_i v^i}{c_n} e_n
\end{aligned}$$

where v_0, being a linear combination of the basis vectors f_i, is in the kernel of f.

Now supposing $\ker g = \ker f$ and writing $g = \sum d_i \alpha^i$, we have that

$$g(v) = g(v_0) + \sum_{i=1}^{n} \frac{c_i v^i}{c_n} g(e_n) = \sum_{i=1}^{n} \frac{c_i v^i}{c_n} d_n = \sum_{i=1}^{n} \frac{d_n}{c_n} c_i v^i = \frac{d_n}{c_n} f(v)$$

Exercise 3.4: A characterization of alternating k-tensors Let f be a k-tensor on a vector space V. Prove that f is alternating if and only if f changes sign whenever two successive arguments are interchanged.

$$f(\ldots, v_{i+1}, v_i, \ldots) = -f(\ldots, v_i, v_{i+1}, \ldots)$$

for $i = 1, \ldots, k-1$.

Solution: We show that any tranposition $(n\ m)$ can be written as the product of an odd number of transpositions of the form $(i\ i+1)$. If $m > n+1$ then

$$(n\ m) = (n\ n+1)(n+1\ m)(n\ n+1)$$

since this sends $n \mapsto n+1 \mapsto m$, $m \mapsto n+1 \mapsto n$, $n+1 \mapsto n \mapsto n+1$ and no other elements are mentioned. By induction on $m-n$ we have that $(n+1\ m)$ is a product of an odd number of transpositions of the form $(i\ i+1)$ so that $(n\ m)$ is too. The base case $(n\ n+1)$ is trivial.

Then, writing an arbitrary permutation τ as a product of k transpositions $\prod_{i=1}^{k}(n_i\ m_i)$ we have that $\operatorname{sgn}\tau = -1^k$. By writing each $(n_i\ m_i)$ as a product of $2k_i + 1$ transpositions of the form $(i\ i+1)$ we have that τ is a product of a total of

$$\sum_{i=1}^{k}(2k_i + 1) = 2\left(\sum_{i=1}^{k} k_i\right) + k$$

such transpositions.

Now, assuming that each such transposition flips the sign of f and letting $j = \sum_i k_i$, we have that

$$\begin{aligned}\tau f &= -1^{2j+k} f \\ &= -1^k f \\ &= (\operatorname{sgn}\tau) f\end{aligned}$$

so that f is alternating.

Conversely, if f is alternating then the sign is flipped by successive interchange since this is the transposition $(i\ i+1)$ and $\operatorname{sgn}((i\ i+1)) = -1$.

Exercise 3.5: Another characterization of alternating k-tensors

Let f be a k-tensor on a vector space V. Prove that f is alternating if and only if $f(v_1, \ldots, v_k) = 0$ whenever two of the vectors $v_1, \ldots, v_k$ are equal.

If f is alternating and $v_i = v_j$ for two of the vectors then

$$\begin{aligned} f(v_1, \dots, v_i, \dots, v_j, \dots, v_k) &= -f(v_1, \dots, v_j, \dots, v_i, \dots, v_k) \\ &= -f(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \end{aligned}$$

so that $2f(v_1, \dots, v_k) = 0$ and thus $f(v_1, \dots, v_k) = 0$ after dividing by 2.

Conversely, we have

$$\begin{aligned} &f(v_1, \dots, v_i, \dots, v_j, \dots, v_k) + f(v_1, \dots, v_j, \dots, v_i, \dots, v_k) \\ &\qquad = f(2v_1, \dots, v_i + v_j, \dots, v_j + v_i, \dots, 2v_k) \\ &\qquad = 0 \end{aligned}$$

since $v_i+v_j = v_j+v_i$. Thus for any transposition τ, $\tau f(v_1, \dots, v_k) = -1f(v_1, \dots, v_k)$ and writing an arbitrary permutation σ as a product of k transpositions we have $\sigma f = -1^k f = (\operatorname{sgn} \sigma) f$ so that f is alternating.

This is sufficient to show that f is alternating if and only if $f(v_1, \dots, v_k) = 0$ whenever $v_1, \dots, v_k$ are linearly dependent. Thus assume that f is alternating and that, without loss of generality, that $v_k = \sum_{j<k} a^j v_j$. Then

$$\begin{aligned} f(v_1, \dots, v_k) &= f(v_1, \dots, \sum_{j<k} a^j v_j) \\ &= \sum_{j<k} a^j f(v_1, \dots, v_j, \dots, v_j) \\ &= 0 \end{aligned}$$

since every term in the sum is zero by the solution to this exercise.

Conversely, use the same argument as above since equality (in this case of $v_i + v_j = v_j + v_i$) is a linear dependence.

Exercise 3.6: Wedge Product and Scalars

Let V be a vector space. For $a, b \in \mathbb{R}$, $f \in A_k(V)$, $g \in A_l(V)$), show that $af \wedge bg = (ab)f \wedge g$.

Solution: By definition

$$\begin{aligned} af \wedge bg(v_1, \dots, v_{k+l}) &= \sum_\sigma (\operatorname{sgn} \sigma) af(v_{\sigma(1)}, \dots, v_{\sigma(k)}) bg(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}) \\ &= ab \sum_\sigma (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}) \\ &= (ab) f \wedge g(v_1, \dots, v_{k+l}) \end{aligned}$$

where σ ranges over all (k, l)-shuffles.

Exercise 3.7: Transformation rule for a wedge product of covectors

Suppose two sets of covectors on a vector space V, $\beta^1, \dots, \beta^l$ and $\gamma^1, \dots, \gamma^l$, are related by

$$\beta^i = \sum_{j=1}^{l} \alpha^i_j \gamma^j, \quad i = 1, \dots, l$$

for a $k \times k$ matrix $A = [\alpha^i_j]$. Show that

$$\beta^1 \wedge \ldots \wedge \beta^l = (\det A) \gamma^1 \wedge \ldots \wedge \gamma^l$$

Solution: By Proposition 3.27, given arbitrary vectors $v_1, \dots, v_l$ we have that

$$\bigwedge_{i=1}^{l} \beta^i(v_1, \dots, v_l) = \det\left[\beta^i(v_k)\right]$$

Since $\beta^i(v_k) = \sum_{j=1}^{l} \alpha^i_j \gamma^j(v_k)$ we have that

$$\begin{aligned}\bigwedge_{i=1}^{l} \beta^i(v_1, \dots, v_l) &= \det\left[\sum_{j=1}^{l} \alpha^i_j \gamma^j(v_k)\right] \\ &= \det\left(\left[\alpha^i_j\right]\left[\gamma^j(v_k)\right]\right) \\ &= \det A \det\left[\gamma^j(v_k)\right]\end{aligned}$$

and since $\det\left[\gamma^j(v_k)\right] = \bigwedge_{j=1}^{l} \gamma^j(v_1, \dots, v_l)$ again by Proposition 3.27, we have that

$$\beta^1 \wedge \ldots \wedge \beta^l = (\det A) \gamma^1 \wedge \ldots \wedge \gamma^l$$

since the chosen vectors were arbitrary.

Exercise 3.8: Transformation rule for k-covectors

Let f be a k-covector on a vector space V. Suppose two sets of vectors $u_1, \dots, u_k$ and $v_1, \dots, v_k$ in V are related by

$$u_j = \sum_{i=1}^{k} a^i_j v_i, \quad j = 1, \dots, k$$

for a $k \times k$ matrix $A = [a^i_j]$. Show that

$$f(u_1, \dots, u_k) = (\det A) f(v_1, \dots, v_k)$$

Solution:

$$\begin{aligned}f(u_1, \dots, u_k) &= f\left(\sum_{i=1}^{k} a^i_1 v_i, \dots, \sum_{i=1}^{k} a^i_k v_i\right) \\ &= \sum_{i_1=1}^{k} \cdots \sum_{i_k=1}^{k} a^{i_1}_1 \cdots a^{i_k}_k f(v_{i_1}, \dots, v_{i_k})\end{aligned}$$

Since f is alternating, any term in the above sum where $i_l = i_m$ (so that $v_{i_l} = v_{i_m}$) is zero by Exercise 3.5. Thus the only terms that remain are those for which all of the $i_1, \dots, i_k$ are distinct, which is to say a permutation σ of $1, \dots, k$. In that case, and because f is alternating, the above sum becomes

$$\begin{aligned}\sum_{\sigma \in S_k} a_1^{\sigma(1)} \cdots a_k^{\sigma(k)} f\left(v_{\sigma(1)}, \dots, v_{\sigma(k)}\right) &= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) a_1^{\sigma(1)} \cdots a_k^{\sigma(k)} f(v_1, \dots, v_k) \\ &= (\det A^{\mathsf{T}})\, f(v_1, \dots, v_k) \\ &= (\det A) f(v_1, \dots, v_k)\end{aligned}$$

since the determinant of the transpose of a matrix is the same.

Exercise 4.9: (A closed 1-form on the punctured plane.) Define a 1-form ω on $\mathbb{R}^2 - \{\mathbf{0}\}$ by

$$\omega = \frac{1}{x^2 + y^2}\,(-y dx + x dy)$$

Show that ω is closed.

Solution:

$$\begin{aligned} d\omega &= d\left(\frac{1}{x^2+y^2}\,(-ydx + xdy)\right) \\ &= \frac{-\left(x^2+y^2\right) - (-y)2y}{\left(x^2+y^2\right)^2} dy \wedge dx + \frac{x^2+y^2 - x2x}{\left(x^2+y^2\right)^2} dx \wedge dy \\ &= \frac{-x^2+y^2}{\left(x^2+y^2\right)^2} dy \wedge dx + \frac{-x^2+y^2}{\left(x^2+y^2\right)^2} dx \wedge dy \\ &= \frac{x^2 - y^2 - x^2 + y^2}{\left(x^2+y^2\right)^2} dx \wedge dy \\ &= 0 \end{aligned}$$

Chapter 2

Exercise 6.3: Group of automorphisms of a vector space.

Let V be a finite-dimensional vector space over $\mathbb{R}$, and $\mathrm{GL}(V)$ the group of all linear automorphisms of V. Relative to an ordered basis $(e_1, \ldots, e_n)$ for V, a linear automorphism for $L \in \mathrm{GL}(V)$ is represented by a matrix $[a^i_j]$ defined by

$$Le_j = \sum_i a^i_j e_i$$

The map

$$\begin{aligned} \phi_e : \mathrm{GL}(V) &\to \mathrm{GL}(n, \mathbb{R}) \\ L &\mapsto \left[a^i_j\right] \end{aligned}$$

is a bijection with an open subset of $\mathbb{R}^{n\times n}$ that makes $\mathrm{GL}(V)$ into a C^∞ manifold, which we denote temporarily by $\mathrm{GL}(V)_e$. If $\mathrm{GL}(V)_u$ is the manifold structure induced from another ordered basis $(u_1, \ldots, u_n)$ for V, show that $\mathrm{GL}(V)_e$ is the same as $\mathrm{GL}(V)_u$.

Solution: Note that if α^i is the dual basis to e_i then $\alpha^i L e_j = a^i_j$. Let β^i be the dual basis to u_i and let T be the linear automorphism defined by $Te_i = u_i$. Then we have that $T^{-1}u_i = e_i$ so that $\alpha^i T^{-1} u_j = \alpha^i e_j = \delta^i_j$. This demonstrates that $\beta^i = \alpha^i T^{-1}$. Let $[b^i_j]$ be the matrix representing L relative to u_i. Then $b^i_j = \beta^i L u_j = \alpha^i T^{-1} L T e_j$. Let T and T^{-1} be represented by $[\tau^i_j]$ and $[\sigma^i_j]$ respectively, relative to e_i. By multiplying matrix entries, we then have that

$$b^i_j = \sum_{k,l} \sigma^i_k a^k_l \tau^l_j$$

showing that each b^i_j is a linear function of the coordinates a^i_j. This shows that these two charts are compatible and thus belong to the same maximal atlas.

Chapter 3

Exercise 8.1: Differential of a map

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the map

$$(u, v, w) = F(x, y) = (x, y, xy)$$

Let $p = (x, y) \in \mathbb{R}^2$. Compute $F_*\left(\partial/\partial x|_p\right)$ as a linear combination of $\partial/\partial u$, $\partial, \partial v$, and $\partial/\partial w$ at $F(p)$.

Solution: Following Proposition 8.11, F_* at p corresponds to the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ y & x \end{bmatrix}$$

relative to the bases $\partial/\partial x|_p$, $\partial/\partial y|_p$ and $\partial/\partial u|_{F(p)}$, $\partial/\partial v|_{F(p)}$, $\partial/\partial w|_{F(p)}$. Thus,

$$F_*\left(\frac{\partial}{\partial x}\bigg|_p\right) = \frac{\partial}{\partial u}\bigg|_{F(p)} + y\frac{\partial}{\partial w}\bigg|_{F(p)}$$

and we can also easily compute

$$F_*\left(\frac{\partial}{\partial y}\bigg|_p\right) = \frac{\partial}{\partial v}\bigg|_{F(p)} + x\frac{\partial}{\partial w}\bigg|_{F(p)}$$

Exercise 8.2: Differential of a linear map

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. For any $p \in \mathbb{R}^n$, there is a canonical identification $T_p\mathbb{R}^n \to \mathbb{R}^n$ given by

$$\sum a^i \frac{\partial}{\partial x^i}\bigg|_p \mapsto \mathbf{a} = \langle a^1, \dots, a^n \rangle$$

Show that the differential $L_* : T_p\mathbb{R}^n \to T_{L(p)}\mathbb{R}^m$ is the map $L : \mathbb{R}^n \to \mathbb{R}^m$ itself with the identification of the tangent spaces as above.

Solution: We prove the following more general proposition:

Theorem 3.1. *Any linear map $L : T_p\mathbb{R}^n \to T_q\mathbb{R}^m$ corresponds, under the identification above, to a unique affine map $A : \mathbb{R}^n \to \mathbb{R}^m$ with $A(p) = q$ such that $A_* = L$. Furthermore, A is an isomorphism if and only if L is.*

This solves the exercise since any linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ is affine so that L will equal the map A of the proposition.

Proof. Let $L : T_p\mathbb{R}^n \to T_q\mathbb{R}^m$ and let x^i be the standard coordinates on $\mathbb{R}^n$ and $\mathbb{R}^m$. Then we have that

$$L\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \sum_{i=1}^m a^i_j \frac{\partial}{\partial x^i}\bigg|_q$$

so that $L = \left[a^i_j\right]$ relative to the bases $\partial/\partial x^i\big|_p$ and $\partial/\partial x^i\big|_q$.

Given $p = \langle p^1, \dots, p^n\rangle$ and $q = \langle q^1, \dots, q^m\rangle$, let $o \in \mathbb{R}^m$ be defined by $o^i = q^i - \sum_j a^i_j p^j$ and define $A : \mathbb{R}^n \to \mathbb{R}^m$ by

$$A^i\left(x^1, \cdots, x^n\right) = o^i + \sum_{j=1}^{n} a^i_j x^j$$

In other words, $A(x) = o + \bar{L}(x)$ where $\bar{L} : \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation determined by the same matrix $\left[a^i_j\right]$ as L, but here relative to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$.

By Proposition 8.11 we have that

$$\begin{aligned}
A_*\left(\frac{\partial}{\partial x^j}\bigg|_p\right) &= \sum_{i=1}^{m} \frac{\partial A^i(x^1, \dots, x^n)}{\partial x^j}\, \frac{\partial}{\partial x^i}\bigg|_q \\
&= \sum_{i=1}^{m} \frac{\partial\left(o^i + \sum_{k=1}^n a^i_k x^k\right)}{\partial x^j}\, \frac{\partial}{\partial x^i}\bigg|_q \\
&= \sum_{i=1}^{m}\sum_{k=1}^{n} a^i_k \frac{\partial x^k}{\partial x^j}\, \frac{\partial}{\partial x^i}\bigg|_q \\
&= \sum_{i=1}^{m} a^i_j\, \frac{\partial}{\partial x^i}\bigg|_q
\end{aligned}$$

so that $A_* = L$. Furthermore

$$A^i(p) = o^i + \sum_j a^i_j p^j = q^i - \sum_j a^i_j p^j + \sum_j a^i_j p^j = q^i$$

so that $A(p) = q$.

Finally if L is an isomorphism then $A^{-1}(y) = -\bar{L}^{-1}(o) + \bar{L}^{-1}(y)$ since

$$\begin{aligned}
A^{-1}\left(A(x)\right) &= A^{-1}\left(o + \bar{L}(x)\right) \\
&= -\bar{L}^{-1}(o) + \bar{L}^{-1}\left(o + \bar{L}(x)\right) \\
&= -\bar{L}^{-1}(o) + \bar{L}^{-1}(o) + \bar{L}^{-1}\left(\bar{L}(x)\right) \\
&= x
\end{aligned}$$

and

$$\begin{aligned}
A\left(A^{-1}(y)\right) &= A\left(-\bar{L}^{-1}(o) + \bar{L}^{-1}(y)\right) \\
&= o + \bar{L}\left(-\bar{L}^{-1}(o) + \bar{L}^{-1}(y)\right) \\
&= o - \bar{L}\left(\bar{L}^{-1}(o)\right) + \bar{L}\left(\bar{L}^{-1}(y)\right) \\
&= y
\end{aligned}$$

I still have to prove the other direction, that A is unique and that L is an isomorphism if A is. $\square$

Exercise 8.3: Differential of a linear map

Fix a real number α and define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{bmatrix} u \\ v \end{bmatrix} = (u, v) = F(x, y) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let $X = -y\partial/\partial x + x\partial/\partial y$ be a vector field on $\mathbb{R}^2$. If $p = (x, y) \in \mathbb{R}^2$ and $F_*(X_p) = (a\partial/\partial u + b\partial/\partial v)\big|_{F(p)}$, find a and b in terms of x, y and α.

Solution: From Exercise 8.2, because F is linear, F_* at p is represented by the same matrix so that

$$\begin{bmatrix} a \\ b \end{bmatrix} = F_*\left(-y\frac{\partial}{\partial x}\bigg|_p + x\frac{\partial}{\partial y}\bigg|_p \right) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} -y \\ x \end{bmatrix}$$

so that $a = -y\cos\alpha - x\sin\alpha$ and $b = -y\sin\alpha + x\cos\alpha$.

Exercise 8.4: Transition matrix for coordinate vectors

Let x, y by the standard coordinates on $\mathbb{R}^2$ and let U be the open set

$$U = \mathbb{R}^2 - \{(x, 0) \mid x \geq 0\}$$

On U the polar coordinates (r, θ) are uniquely defined by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, \quad r > 0,\ 0 < \theta < 2\pi \end{aligned}$$

Find $\partial/\partial r$ and $\partial/\partial\theta$ in terms of $\partial/\partial x$ and $\partial/\partial y$.

Solution:

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} \\ &= \frac{\partial(r\cos\theta)}{\partial r}\frac{\partial}{\partial x} + \frac{\partial(r\sin\theta)}{\partial r}\frac{\partial}{\partial y} \\ &= \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \end{aligned}$$

and similarly

$$\begin{aligned} \frac{\partial}{\partial\theta} &= \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} \\ &= \frac{\partial(r\cos\theta)}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial(r\sin\theta)}{\partial\theta}\frac{\partial}{\partial y} \\ &= -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} \end{aligned}$$

Exercise 8.5: Velocity of a curve in local coordinates

Prove Proposition 8.15.

Solution: Given a chart (U, x^i) we have that

$$\begin{aligned} dx^i\left(c'(t_0)\right) &= c_*\left(\left.\frac{d}{dt}\right|_{t_o}\right)(x^i) \\ &= \left.\frac{d}{dt}\right|_{t_o}(x^i \circ c)(t) \\ &= \left.\frac{d}{dt}\right|_{t_o} c^i(t) \\ &= \dot{c}^i(t_0) \end{aligned}$$

so that $c'(t_0) = \sum \dot{c}^i(t_0)\frac{\partial}{\partial x^i}$.

Exercise 8.6: Velocity vector Let $p = (x, y)$ be a point in $\mathbb{R}^2$. Then

$$c_p(t) = \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \quad t \in \mathbb{R}$$

is a curve with initial point p in $\mathbb{R}^2$. Compute the velocity vector $c_p'(0)$.

Solution: In the basis $\partial/\partial x$, $\partial/\partial y$ we have that

$$c_p'(t) = \begin{bmatrix} -2\sin 2t & -2\cos 2t \\ 2\cos 2t & -2\sin 2t \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \quad t \in \mathbb{R}$$

so that

$$c_p'(0) = -2y\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y}$$

Exercise 8.7: Tangent space to a product

If M and N are manifolds, let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be the two projections. Prove that for $(p, q) \in M \times N$,

$$(\pi_{1*}, \pi_{2*}) : T_{(p,q)}(M \times N) \to T_pM \times T_qN$$

is an isomorphism.

Solution: Let (U_p, ϕ) and (U_q, ψ) be charts around $p \in M$ and $q \in N$ respectively. Then by definition $\left(U_p \times U_q, (\phi, \psi)\right)$ is a chart around $(p, q) \in M \times N$. Letting $x^i = r^i \circ \phi$, $y^i = r^i \circ \psi$, and $z^i = r^i \circ (\phi, \psi)$ be the coordinates on U_p and U_q and $U_p \times U_q$ we have that $\partial/\partial x^i$, $\partial/\partial y^i$ and $\partial/\partial z^i$ are bases for T_pM, T_qN and $T_{(p,q)}(M \times N)$. Notice that

$$\begin{aligned} z^i &= x^i \circ \pi_1 = \pi_1^i \quad 1 \le i \le m \\ z^{m+i} &= y^i \circ \pi_2 = \pi_2^i \quad 1 \le i \le n \end{aligned}$$

so that by Proposition 8.11 we have

$$\begin{aligned}
\pi_{1*}\left(\left.\frac{\partial}{\partial z^j}\right|_{(p,q)}\right) &= \sum_{i=1}^{m} \frac{\partial \pi_1^i}{\partial z^j}(p,q) \left.\frac{\partial}{\partial x^i}\right|_p \\
&= \sum_{i=1}^{m} \frac{\partial z^i}{\partial z^j}(p,q) \left.\frac{\partial}{\partial x^i}\right|_p
\end{aligned}$$

and thus

$$\begin{aligned}
\pi_{1*}\left(\left.\frac{\partial}{\partial z^j}\right|_{(p,q)}\right) &= \left.\frac{\partial}{\partial x^j}\right|_p \qquad 1 \le j \le m \\
\pi_{1*}\left(\left.\frac{\partial}{\partial z^{m+j}}\right|_{(p,q)}\right) &= 0 \qquad 1 \le j \le n
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\pi_{2*}\left(\left.\frac{\partial}{\partial z^j}\right|_{(p,q)}\right) &= \sum_{i=1}^{n} \frac{\partial \pi_2^i}{\partial z^j}(p,q) \left.\frac{\partial}{\partial y^i}\right|_q \\
&= \sum_{i=1}^{n} \frac{\partial z^{m+i}}{\partial z^j}(p,q) \left.\frac{\partial}{\partial y^i}\right|_q
\end{aligned}$$

so that

$$\begin{aligned}
\pi_{2*}\left(\left.\frac{\partial}{\partial z^j}\right|_{(p,q)}\right) &= 0 \qquad 1 \le j \le m \\
\pi_{2*}\left(\left.\frac{\partial}{\partial z^{m+j}}\right|_{(p,q)}\right) &= \left.\frac{\partial}{\partial y^j}\right|_q \qquad 1 \le j \le n
\end{aligned}$$

Thus

$$(\pi_{1*}, \pi_{2*})\left(\sum_{i=1}^{m+n} a^i \frac{\partial}{\partial z^i}\right) = \left(\sum_{i=1}^{m} a^i \frac{\partial}{\partial x^i}, \sum_{i=1}^{n} a^{m+i} \frac{\partial}{\partial y^i}\right)$$

which is a bijection since $(\partial/\partial x^i, 0), (0, \partial/\partial y^i)$ is a basis for $T_pM \times T_qN$.

Now we consider the image of a velocity vector under this isomorphism. Any curve $c :]a,b[\to M \times N$ starting at (p,q) can be written $c(t) = ((c_1(t), c_2(t))$ where $c_1 = \pi_1 \circ c$ and $c_2 = \pi_2 \circ c$. Then c_1 and c_2 are curves in M and N starting at p and q respectively and by Proposition 8.18 we have that

$$\begin{aligned}
(\pi_{1*}, \pi_{2*})\,(c'(0)) &= (\pi_{1*}(c'(0)), \pi_{2*}(c'(0))) \\
&= ((\pi_1 \circ c)'(0), (\pi_2 \circ c)'(0)) \\
&= (c_1'(0), c_2'(0))
\end{aligned}$$

Exercise 8.8: Differentials of mutliplication and inverse

Let G be a Lie group with multiplication map $\mu : G \times G \to G$, inverse map $\iota : G \to G$ and identity element e.

(i) Show that the differential at the identity of the multiplication map μ is addition:

$$\mu_* : T_eG \times T_eG \to T_eG$$
$$\mu_*(X_e, Y_e) = X_e + Y_e$$

(ii) Show that the differential at the identity of ι is the negative

$$\iota_* : T_eG \to T_eG$$
$$\iota_*(X_e) = -X_e$$

Solution:

(i) To any curve $c :]a, b[:\to G$ at e corresponds the curve $\bar{c}(t) = (c(t), e) \in G \times G$ at (e, e). Given the previous exercise we have that $\bar{c}'(0) = (c'(0), \mathbf{0})$ since e is a constant. Since e is the identity, we have that $\mu(c(t), e) = c(t)$ which is to say that $\mu \circ \bar{c} = c$. Then we have by Proposition 8.18 that

$$\mu_* \left(c'(0), \mathbf{0}\right) = \mu_* \left(\bar{c}'(0)\right) = (\mu \circ \bar{c})'(0) = c'(0)$$

and similarly

$$\mu_* \left(\mathbf{0}, c'(0)\right) = c'(0)$$

Since μ_* is linear we have that

$$\begin{aligned}\mu_* \left(c_1'(0), c_2'(0)\right) &= \mu_* \left((c_1'(0), \mathbf{0}) + (\mathbf{0}, c_2'(0))\right) \\ &= \mu_* \left(c_1'(0), \mathbf{0}\right) + \mu_* \left(\mathbf{0}, c_2'(0)\right) \\ &= c_1'(0) + c_2'(0)\end{aligned}$$

Since a velocity vector $c'(0)$ can represent any vector $X_e \in T_eG$, we have that μ_* is addition.

(ii) To any curve $c(t) \in G$ at e, associate the curve $\bar{c}(t) = (c(t), \iota(c(t)))$ so that $\bar{c}'(0) = (c'(0), (\iota \circ c)'(0)))$. Because ι is the inverse, we have that $\mu(\bar{c}(t)) = e$ is constant so that $(\mu \circ \bar{c})'(0) = \mathbf{0}$. Again by proposition 8.18, we have

$$\begin{aligned}\mu_* \left(c'(0), \iota_* \left(c'(0)\right)\right) &= \mu_* \left(c'(0), (\iota \circ c)'(0)\right) \\ &= \mu_* \left(\bar{c}'(0)\right) \\ &= (\mu \circ \bar{c})'(0) \\ &= \mathbf{0}\end{aligned}$$

Since μ_* is addition and $c'(0)$ can represent any vector X_e we have that

$$X_e + \iota_*(X_e) = \mathbf{0}$$

so that $\iota_*(X_e) = -X_e$.

Exercise 8.9: Transforming vectors to coordinate vectors

Let $X_1, \dots, X_n$ be n vector fields on an open subset U of a manifold of dimension n. Suppose that at $p \in U$, the vectors $X_1(p), \dots, X_n(p)$ are linearly independent. Show that there is a chart $(V, x^1, \dots, x^n)$ about p such that $X_i(p) = \partial/\partial x^i|_p$ for $i = 1, \dots, n$.

Solution: Let $(V, \psi) = (V, y^1, \dots, y^n)$ be a coordinate chart around p and let $L : T_pV \to T_pV$ be the linear transformation defined by

$$L\left(\left.\frac{\partial}{\partial y^j}\right|_p \right) = X_j(p)$$

which is an isomorphism since the $X_j(p)$ form a basis. Then by Exercise 8.2, $\psi_* \circ L \circ \psi_*^{-1} : T_{\psi(p)}\mathbb{R}^n \to T_{\psi(p)}\mathbb{R}^n$ corresponds uniquely to an affine transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ with $A(\psi(p)) = \psi(p)$ such that $A_* = \psi_* \circ L \circ \psi_*^{-1}$. Equivalently,

$$L = \psi_*^{-1} \circ A_* \circ \psi_* = \left(\psi^{-1} \circ A \circ \psi\right)_* \tag{2}$$

Since L is an isomorphism, so is A so that $A^{-1} \circ \psi$ is a diffeomorphism. By Proposition 6.11, $\left(V, A^{-1} \circ \psi\right) = \left(V, x^1, \dots, x^n\right)$ is a chart in the differentiable structure of the manifold. By Proposition 8.8,

$$\left(A^{-1} \circ \psi\right)_* \left(\left.\frac{\partial}{\partial x^j}\right|_p \right) = \left.\frac{\partial}{\partial r^j}\right|_{\psi(p)}$$

so that by functoriality

$$\left.\frac{\partial}{\partial x^j}\right|_p = \left(A^{-1} \circ \psi\right)_*^{-1} \left(\left.\frac{\partial}{\partial r^j}\right|_{\psi(p)} \right) = \left(\psi^{-1} \circ A\right)_* \left(\left.\frac{\partial}{\partial r^j}\right|_{\psi(p)} \right) \tag{3}$$

Then, again by Proposition 8.8 and using (2) and (3), we now have that

$$\begin{aligned}
X_j(p) &= L\left(\left.\frac{\partial}{\partial y^j}\right|_p \right) \\
&= \left(\psi^{-1} \circ A \circ \psi\right)_* \left(\left.\frac{\partial}{\partial y^j}\right|_p \right) \\
&= \left(\psi^{-1} \circ A\right)_* \left(\left.\frac{\partial}{\partial r^j}\right|_{\psi(p)} \right) \\
&= \left.\frac{\partial}{\partial x^j}\right|_p
\end{aligned}$$

Exercise 9.1: Regular Values

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = x^3 - 6xy + y^2$$

Find all values $c \in \mathbb{R}$ for which the level set $f^{-1}(c)$ is a regular submanifold of $\mathbb{R}^2$.

Solution: The Jacobian of f is

$$\begin{bmatrix} 3x^2 - 6xy & 2y - 6x \end{bmatrix}$$

which has rank 0 when

$$\begin{aligned} 3x^2 - 6y &= 0 \\ 2y - 6x &= 0 \end{aligned}$$

which has solutions at

$$\begin{aligned} x &= 0, y = 0 \\ x &= 6, y = 18 \end{aligned}$$

These values belong to the level sets

$$\begin{aligned} 0^3 - 6 \cdot 0 \cdot 0 + 0^2 &= 0 \\ 6^3 - 6 \cdot 6 \cdot 18 + 18^2 &= -108 \end{aligned}$$

In Figure 1, a red $\times$ marks the two critical points. Also shown are the level sets at $-300, -100, 0$ and 50. Level sets greater than 0, such as at 50, are composed of one connected component. At 0 a loop forms around $(6, 18)$. At -108 this loop converges to a point and below -108 the loop component disappears and the sub-manifold is one connected component again such as the level set at -300 in the figure.

Exercise 9.3: Solution set of two equations

Is the solution set of the system of equations

$$x^3 + y^3 + z^3 = 1, \quad z = xy$$

in $\mathbb{R}^3$ a smooth manifold? Prove your answer.

Solution: The solution set of these equations corresponds to the zero set of the map $\mathbb{R}^3 \to \mathbb{R}^2$ defined by $(x, y, z) \mapsto (x^3 + y^3 + z^3 - 1, z - xy)$. This map's Jacobian is

$$\begin{bmatrix} 3x^2 & 3y^2 & 3z^2 \\ -y & -x & 1 \end{bmatrix}$$

which has rank less than 2 when all 2×2 minors have determinant zero. This corresponds to the equations

$$-3x^3 + 3y^3 = 0 \tag{4}$$

$$3x^2 + 3yz^2 = 0 \tag{5}$$

$$3y^2 + 3xz^2 = 0 \tag{6}$$

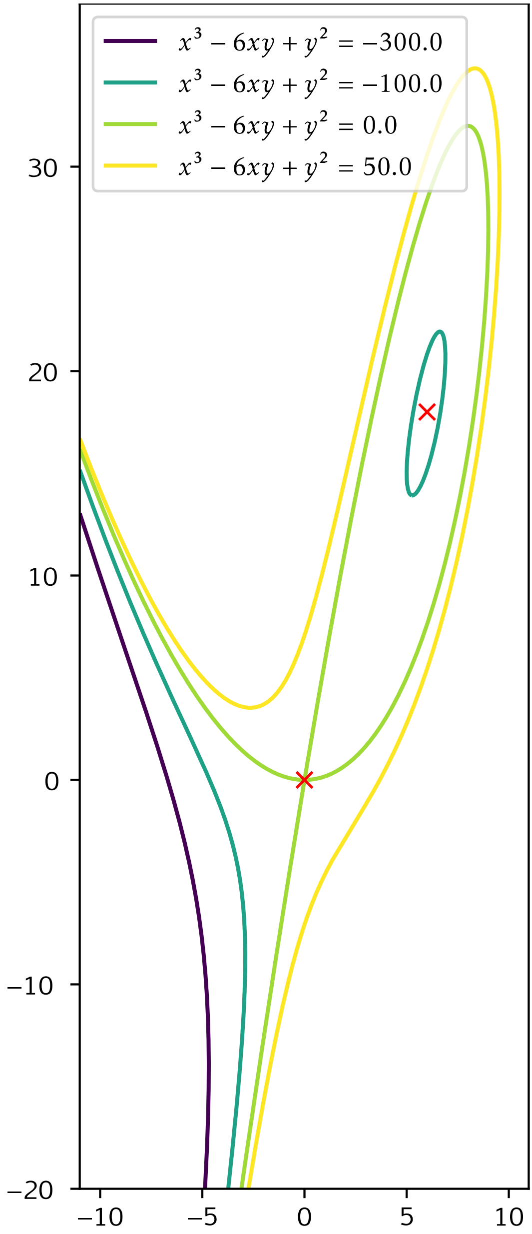

Figure 1: Level sets for $f(x, y) = x^3 - 6xy + y^2$

Equation (4) gives that $x = y$ so that equations (5) and (6) both become

$$0 = 3x^2 + 3xz^2 = 3x(x + z^2)$$

and thus either $x = y = 0$ and z can be any value or $x = y = -z^2$. In other words, the potential critical points of this system of equations lie in the image of the two curves $\mathbb{R} \to \mathbb{R}^3$ defined by

$$z \mapsto (0, 0, z) \tag{7}$$

$$z \mapsto \left(-z^2, -z^2, z\right) \tag{8}$$

In case (7), z must also be 0 since $z = xy = 0 \cdot 0 = 0$ but $(0, 0, 0)$ does not satisfy $x^3 + y^3 + z^3 = 1$. In case (8), $z = xy = z^2z^2 = z^4$ so that $z = 1$ and therefore $x = y = -1$. But the point $(-1, -1, 1)$ also does not satisfy $x^3 + y^3 + z^3 = 1$ so that no potential critical points satisfy the system of equations. Thus the solution set is a regular manifold by Theorem 9.9.

Figure 2 shows the two surfaces with their intersection drawn on each. Figure 3 shows only the curve

Exercise 9.4: Regular submanifolds

Suppose that a subset S of $\mathbb{R}^2$ has the property that locally on S one of the coordinates is a C^∞ function of the other coordinate. Show that S is a regular submanifold of $\mathbb{R}^2$.

Solution: Let $p \in S$ be such that there is a neighborhood $p \in U$ and a function $f : U_x \to U_y$ such that $S \cap U = \{(x, y) \mid y = f(x)\}$. Notice that the map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $(x, y) \mapsto (x, y - f(x))$ is a diffeomorphism since it has smooth inverse $(u, v) \mapsto (u, v + f(u))$. Since $S \cap U$ is the vanishing of $y - f(x)$, $(U, (x, y - f(x)))$ is an adapted chart around p. Since p was arbitrary S is a regular submanifold.

Exercise 9.5: Graph of a smooth function

Show that the graph $\Gamma(f)$ of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\Gamma(f) = \left\{(x, y, f(x, y)) \in \mathbb{R}^3\right\}$$

is a regular submanifold of $\mathbb{R}^3$.

Solution: The function $\phi : \mathbb{R}^3 \mapsto \mathbb{R}^3$ defined by $\phi(x, y, z) = (x, y, z - f(x, y))$ is a diffeomorphism since it has smooth inverse $(u, v, w) \mapsto (u, v, w + f(u, v))$. Since $\Gamma(f)$ is defined by the vanishing of $z - f(x, y)$ this is an adapted chart relative to $\Gamma(f)$ so that it is a regular submanifold.

Exercise 9.6: Euler's Formula

A polynomial $F(x_0, \dots, x_n) \in \mathbb{R}[x_0, \dots, x_n]$ is *homogeneous of degree k* if it is a linear combination of monomials $x_0^{i_0} \cdots x_n^{i_n}$ of degree $\sum_{j=0}^n i_j = k$. Let $F(x_0, \dots, x_n)$ be a homogeneous polynomial of degree k. Clearly, for any $t \in \mathbb{R}$,

$$F(tx_0, \dots, tx_n) = t^k F(x_0, \dots, x_n) \tag{9}$$

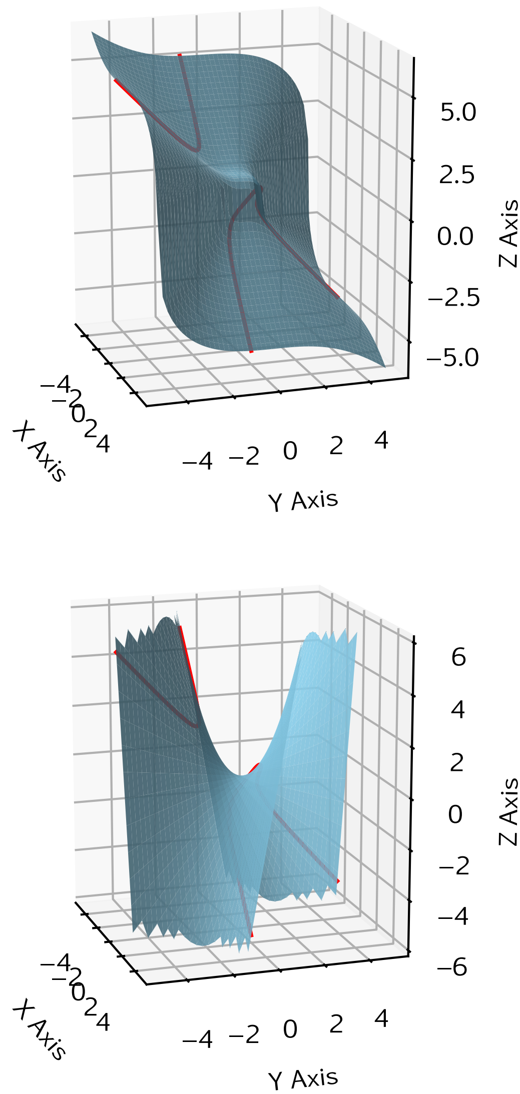

Figure 2: Surfaces $x^3 + y^3 + z^3 = 1$ and $z = xy$ with intersection curve

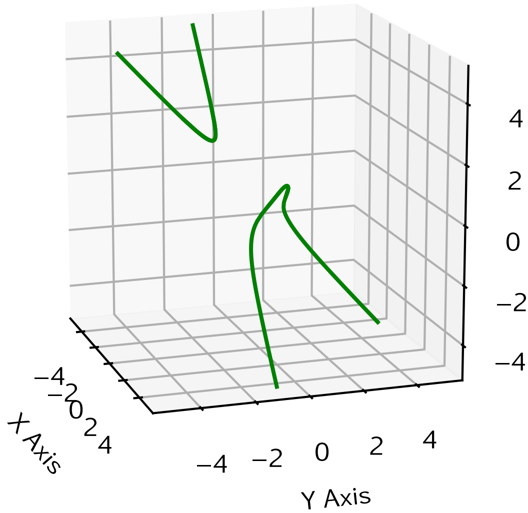

Figure 3: Intersection of $x^3 + y^3 + z^3 = 1$ and $z = xy$

Show that

$$\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = kF \tag{10}$$

Solution: Differentiating both sides of (9) gives that

$$\frac{d}{dt} F(tx_0, \dots, tx_n) = \sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i}(tx_0, \dots, tx_n)$$

and

$$\frac{d}{dt} t^k F(x_0, \dots, x_n) = kt^{k-1} F(x_0, \dots, x_n)$$

so that

$$\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i}(tx_0, \dots, tx_n) = kt^{k-1} F(x_0, \dots, x_n)$$

Choosing $t = 1$ gives the result.

Exercise 9.7: Smooth projective hypersurface

Let $F(x_0, \dots, x_n)$ be a homogeneous polynomial of degree k. Show that the projective hypersurface $Z(F) \in \mathbb{R}P^n$ defined by $F(x_0, \dots, x_n) = 0$ is smooth if the $\partial F / \partial x_i$ are not all simultaneously zero on $Z(F)$.

Solution: Let $[q] \in Z(F)$ and assume without loss of generality that

$$[q] \in U_0 = \{[x_0, \dots, x_n] \mid x_0 \neq 0\}$$

so we can write $[q] = [1, p_1, \dots, p_n] = [\iota_0(p)]$ where $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ and $\iota_0(p_1, \dots, p_n) = (1, p_1, \dots, p_n)$.

Define $f_0 = F \circ \iota_0 : \mathbb{R}^n \to \mathbb{R}$. Then on the chart U_0, ϕ_0 with $\phi_0(x_0, \dots, x_n) = (1, x_1/x_0, \dots, x_n/x_0)$, $Z(F) \cap U_0$ is the zero set of f_0 since

$$F(x_0, \dots, x_n) = x_0^k \cdot F(1, x_1/x_0, \dots, x_n/x_0) = x_0^k \cdot f_0(x_1/x_0, \dots, x_n/x_0)$$

implies that $f_0(x_1/x_0, \dots, x_n/x_0) = 0$ if and only if $F(x_0, \dots, x_n) = 0$ whenever $x_0 \neq 0$. Also notice that by the chain rule we have that

$$\begin{aligned}
\frac{\partial f_0}{\partial x_i}(p) &= \sum_j \frac{\partial F}{\partial x_j}(\iota_0(p)) \frac{\partial \iota_0^j}{\partial x_i}(p) \\
&= \sum_j \frac{\partial F}{\partial x_j}(\iota_0(p)) \frac{\partial x_j}{\partial x_i}(p) \\
&= \frac{\partial F}{\partial x_i}(\iota_0(p))
\end{aligned}$$

for $i = 1, \dots, n$.

Using equation (10) with $x_0 = 1$ we have that

$$\frac{\partial F}{\partial x_0}(\iota_0(p)) + \sum_{i=1}^n p_i \frac{\partial F}{\partial x_i}(\iota_0(p)) = k f_o(p) = 0$$

since $p \in Z(f_0)$. Thus

$$\begin{aligned}
\frac{\partial F}{\partial x_0}(\iota_0(p)) &= -\sum_{i=1}^n p_i \frac{\partial F}{\partial x_i}(\iota_0(p)) \\
&= -\sum_{i=1}^n p_i \frac{\partial f_0}{\partial x_i}(p)
\end{aligned}$$

Thus not all of the $\frac{\partial f_0}{\partial x_i}(p)$ can be zero since that would imply that each

$$\frac{\partial F}{\partial x_i}(\iota_0(p)) = \frac{\partial f_0}{\partial x_i}(p) = 0$$

and also that

$$\frac{\partial F}{\partial x_0}(\iota_0(p)) = -\sum p_i \frac{\partial f_0}{\partial x_i}(p) = -\sum p_i \cdot 0 = 0$$

so that all the $\partial F/\partial x_i$ are zero at $p \in Z(F)$ which we assume not to be the case. Thus by Corollary 9.10, there is an adapted chart $V \subset U_0$ for $Z(F)$ around p. Since p was arbitrary, $Z(F)$ is a regular submanifold of $\mathbb{R}P^n$.

Exercise 11.1: Tangent vectors to a sphere

The unit sphere S^n in $\mathbb{R}^{n+1}$ is defined by the equation $\sum_{i=1}^{n+1}(x^i)^2 = 1$. For $p = (p^1, \dots, p^{n+1}) \in S^n$, show that a necessary and sufficient condition for

$$X_p = \sum a^i \frac{\partial}{\partial x^i}\bigg|_p \in T_p(\mathbb{R}^{n+1})$$

to be tangent to S^n at p is $\sum a^i p^i = 0$.

Solution: Let $c(t) : \mathbb{R} \to \mathbb{R}^{n+1}$ lie in S^n such that $c(0) = p$ and $c'(0) = X_p$. Since $c(t) \in S^n$ we have that $\sum c^i(t)^2 = 1$ and thus by the chain rule

$$\sum_{i=1}^{n+1} 2c^i(t)\dot{c}^i(t) = 0$$

Dividing by 2 and taking $t = 0$ gives $\sum p^i a^i = 0$ so that the condition is necessary. By Exercise 3.2 we know that the hyperplane $H_p = \left\{\sum a^i e_i \mid \sum a^i p^i = 0\right\}$ has dimension n. Thus since $i_*\left(T_p S^n\right) \subset H_p$ also has dimension n, they must be equal so that the condition is sufficient.

Exercise 11.2: Tangent vectors to a plane curve

(i) Let $i : S^1 \hookrightarrow \mathbb{R}^2$ be the inclusion map of the unit circle. In this problem, we denote by x, y the standard coordinates on $\mathbb{R}^2$ and by $\bar{x}, \bar{y}$ their restrictions to S^1. Thus, $\bar{x} = i^*x = x \circ i$ and $\bar{y} = i^*y = y \circ i$. On the upper semicircle $U = \left\{(a, b) \in S^1 \;\middle|\; b > 0\right\}$, $\bar{x}$ is a local coordinate so that $\partial/\partial\bar{x}$ is defined. Prove that for $p \in U$,

$$i_*\left(\frac{\partial}{\partial \bar{x}}\bigg|_p\right) = \left(\frac{\partial}{\partial x} + \frac{\partial \bar{y}}{\partial \bar{x}}\frac{\partial}{\partial y}\right)\bigg|_p$$

Thus, although $i_* : T_p S^1 \to T_P\mathbb{R}^2$ is injective, $\partial/\partial\bar{x}|_p$ cannot be identified with $\partial/\partial x|_p$.

(ii) Generalize (i) to a smooth curve C in $\mathbb{R}^2$, letting U be a chart in C on which $\bar{x}$, the restriction of x to C is a local coordinate.

Solution:

(i) By Proposition 8.11 we have that

$$\begin{aligned}
i_*\left(\frac{\partial}{\partial \bar{x}}\bigg|_p\right) &= \left(\frac{\partial x \circ i}{\partial \bar{x}}(p)\frac{\partial}{\partial x} + \frac{\partial y \circ i}{\partial \bar{x}}(p)\frac{\partial}{\partial y}\right)\bigg|_p \\
&= \left(\frac{\partial \bar{x}}{\partial \bar{x}}(p)\frac{\partial}{\partial x} + \frac{\partial \bar{y}}{\partial \bar{x}}(p)\frac{\partial}{\partial y}\right)\bigg|_p \\
&= \left(\frac{\partial}{\partial x} + \frac{\partial \bar{y}}{\partial \bar{x}}(p)\frac{\partial}{\partial y}\right)\bigg|_p
\end{aligned}$$

In this case since $\bar{y} = \sqrt{1-\bar{x}^2}$, and letting $p = (a, b)$ we have that

$$\begin{aligned} i_*\left(\left.\frac{\partial}{\partial \bar{x}}\right|_p\right) &= \left.\left(\frac{\partial}{\partial x} - \frac{a}{\sqrt{1-a^2}}\frac{\partial}{\partial y}\right)\right|_p \\ &= \left.\left(\frac{\partial}{\partial x} - \frac{a}{b}\frac{\partial}{\partial y}\right)\right|_p \end{aligned}$$

(ii) Assume $C(t_0) = p \in U$ and let $\bar{x} = x \circ C, \bar{y} = y \circ C$ so that

$$C(t) = \left(\bar{x}(t), \bar{y}(t)\right) \tag{11}$$

and

$$C'(t_0) = \left.\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\right|_p$$

where $a = \frac{d\bar{x}}{dt}(t_0)$ and $b = \frac{d\bar{y}}{dt}(t_0)$. Since $\bar{x}$ is a local coordinate, $\bar{x}(t) : \mathbb{R} \to \mathbb{R}$ is a local diffeomorphism in U so that, firstly a is non-zero and secondly there exists a local inverse $\bar{x}^{-1}$ such that $\frac{d\bar{x}^{-1}}{dx}(x_0) = \frac{1}{a}$ where $\bar{x}(t_0) = x_0$. Then by (11)

$$\begin{aligned} \left(C \circ \bar{x}^{-1}\right)(x) &= C(\bar{x}^{-1}(x)) \\ &= \left(\bar{x}(\bar{x}^{-1}(x)), \bar{y}(\bar{x}^{-1}(x))\right) \\ &= \left(x, \left(\bar{y} \circ \bar{x}^{-1}\right)(x)\right) \end{aligned}$$

Then we can say the following: $\left(x, y - \bar{y}(\bar{x}^{-1}(x))\right)$ is an adapted chart for the image of C in $\mathbb{R}^2$ and $C \circ \bar{x}^{-1}$ is the inclusion map relative to the local coordinate $\bar{x}$ of this adapted chart. Thus we have resolved $\bar{x}$ and $\bar{y}$ to be functions of $\bar{x}$ instead of t so that by the chain rule we can now compute

$$\begin{aligned} i_*\left(\left.\frac{\partial}{\partial \bar{x}}\right|_p\right) &= \left.\left(\frac{\partial}{\partial x} + \frac{\partial \bar{y}}{\partial \bar{x}}(x_0)\frac{\partial}{\partial y}\right)\right|_p \\ &= \left.\left(\frac{\partial}{\partial x} + \frac{d\bar{y} \circ \bar{x}^{-1}}{dx}(x_0)\frac{\partial}{\partial y}\right)\right|_p \\ &= \left.\left(\frac{\partial}{\partial x} + \frac{d\bar{y}}{dt}(t_0) \cdot \frac{d\bar{x}^{-1}}{dx}(x_0)\frac{\partial}{\partial y}\right)\right|_p \\ &= \left.\left(\frac{\partial}{\partial x} + \frac{b}{a}\frac{\partial}{\partial y}\right)\right|_p \end{aligned}$$

so that

$$i_*\left(\left.\frac{\partial}{\partial \bar{x}}\right|_p\right) = \frac{1}{a}C'(t_0)$$

thus normalizing the vector $C'(t_0)$ on its first component.

Exercise 11.3: Critical points of a smooth map on a compact manifold

Show that a smooth map f from a compact manifold N to $\mathbb{R}^m$ has a critical point.

Solution: By Proposition A.34, the image of f is compact and therefore closed and bounded by the Heine-Borel Theorem, Theorem A.40. Thus the image of f is a proper subset of $\mathbb{R}^m$. By Proposition A.35, f is a closed map. Now suppose f has no critical points so that f is everywhere a submersion. By Corollary 11.6, f is an open map. Since N is both open and closed, the proper subset $f(N)$ is also open and closed. In particular, $\mathbb{R}^m \setminus f(N)$ is also open so that $\mathbb{R}^m$ is the disjoint union of two nonempty subsets and therefore disconnected. But $\mathbb{R}^m$ is connected so this is a contradiction.

Exercise 11.4: Differential of an inclusion map

On the upper hemisphere of the unit sphere S^2, we have the coordinate map $\phi = (u, v)$, where

$$u(a,b,c) = a \quad \text{and} \quad v(a,b,c) = b$$

So the derivations $\partial/\partial u|_p, \partial/\partial v|_p$ are tangent vectors of S^2 at any point $p = (a,b,c)$ on the upper hemisphere. Let $i : S^2 \to \mathbb{R}^3$ be the inclusion and x, y, z the standard coordinates on $\mathbb{R}^3$. The differential $i_* : T_pS^2 \to T_p\mathbb{R}^3$ maps $\partial/\partial u|p, \partial/\partial v|_p$ into $T_p\mathbb{R}^3$. Thus,

$$i_*\left(\left.\frac{\partial}{\partial u}\right|_p\right) = \alpha^1 \left.\frac{\partial}{\partial x}\right|_p + \beta^1 \left.\frac{\partial}{\partial y}\right|_p + \gamma^1 \left.\frac{\partial}{\partial z}\right|_p$$
$$i_*\left(\left.\frac{\partial}{\partial v}\right|_p\right) = \alpha^2 \left.\frac{\partial}{\partial x}\right|_p + \beta^2 \left.\frac{\partial}{\partial y}\right|_p + \gamma^2 \left.\frac{\partial}{\partial z}\right|_p$$

for some constants $\alpha^i, \beta^i, \gamma^i$. Find $(\alpha^i, \beta^i, \gamma^i)$, for $i = 1, 2$.

Solution: The inverse map $\phi^{-1} = (x, y, z)$ is

$$x(a,b) = a, \quad y(a,b) = b, \quad z(a,b) = \sqrt{1 - a^2 - b^2} = c$$

So that by Proposition 8.11,

$$\begin{aligned}
\alpha^1 &= \frac{\partial x}{\partial u}(a,b) = \frac{\partial u}{\partial u}(a,b) = 1 \\
\alpha^2 &= \frac{\partial y}{\partial u}(a,b) = \frac{\partial v}{\partial u}(a,b) = 0 \\
\alpha^3 &= \frac{\partial z}{\partial u}(a,b) = \frac{\partial \sqrt{1 - u^2 - v^2}}{\partial u}(a,b) = -\frac{a}{\sqrt{1 - a^2 - b^2}} = -\frac{a}{c}
\end{aligned}$$

and similarly

$$\beta^1 = \frac{\partial x}{\partial v}(p) = 0$$
$$\beta^2 = \frac{\partial y}{\partial v}(p) = 1$$
$$\beta^3 = \frac{\partial z}{\partial v}(p) = -\frac{b}{\sqrt{1-a^2-b^2}} = -\frac{b}{c}$$

Exercise 11.5: One-to-one immersion of a compact manifold

Prove that if N is a compact manifold, then a one-to-one immersion $f : N \to M$ is an embedding.

Solution: First note that f restricts to a surjective continuous map onto its image $f : N \to f(N)$ where $f(N)$ is given the relative topology since if $U \cap f(N)$ is an open subset of $f(N)$ for some open set $U \subset M$, then $f^{-1}(U)$ is open in N because f is continuous and $f^{-1}(U) = f^{-1}(U \cap N)$. Because f is one-to-one, it is a continuous bijection to its image. Because N is compact, f is a homeomorphism by Corollary A.35 thus satisfying the second requirement to be an embedding.

Exercise 12.1: Hausdorff condition on the tangent bundle

Prove that the tangent bundle TM of a manifold M is Hausodorff.

Solution: Let $\pi : TM \to M$ be the tangent bundle and let $x_1 \neq x_2 \in TM$. We consider two cases

(i) $\pi(x_1) = \pi(x_2)$
(ii) $\pi(x_1) \neq \pi(x_2)$

In the first case there is a trivializing open set U containing $\pi(x_1) = \pi(x_2) = q$ so that $\phi : E|_U \to U \times \mathbb{R}^m$ is a diffeomorphism. Then we have that $\phi(x_1) = \left(q, \sum c^i(x_1)e_i\right)$ and $\phi(x_2) = \left(q, \sum c^i(x_2)e_i\right)$. Since ϕ is a diffeomorphism, $\phi(x_1) \neq \phi(x_2)$ so that $\sum c^i(x_1)e_i \neq \sum c^i(x_2)e_i$. Because $\mathbb{R}^m$ is Hausdorff, there are disjoint open sets U_1 and U_2 in $\mathbb{R}^m$ containing $\sum c^i(x_1)e_i$ and $\sum c^i(x_2)e_i$ respectively. Then $\phi^{-1}(U \times U_1)$ and $\phi^{-1}(U \times U_2)$ are disjoint open sets containing x_1 andx_2 respectively.

In the second case, there are disjoint open sets U_1 and U_2 in M that contain $\pi(x_1)$ and $\pi(x_2)$ respectively in which case $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ are disjoint open sets that contain x_1 and x_2.

Exercise 12.2: Let $(U, \phi) = (U, x^1, \dots, x^n)$ and $(V, \psi) = (V, y^1, \dots, y^n)$ be overlapping coordinate charts on a manifold M. They induce coordinate charts $(TU, \tilde{\phi})$ and $(TV, \tilde{\psi})$ on the total space TM of the tangent bundle with transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$:

$$(x^1, \dots, x^n, a^1, \dots, a^n) \mapsto (y^1, \dots, y^n, b^1, \dots, b^n)$$

(i) Compute the Jacobian matrix of the transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$ at $\phi(p)$.

(ii) Show that the Jacobian determinant of the transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$ at $\phi(p)$ is $(\det[\partial y^i/\partial x^j])^2$.

Solution: Letting $q = \left(p, \sum_i a^i \frac{\partial}{\partial x^i}\Big|_p\right) \in TU$ we have by definition that

$$\tilde{\psi}^i(q) = y^i(p)$$
$$\tilde{\psi}^{i+n}(q) = \sum_j a^j \frac{\partial y^i}{\partial x^j}(p)$$

for $i = 1, \ldots, n$ in which case the matrix entries of the Jacobian are

$$\frac{\partial \tilde{\psi}^i}{\partial x^j}(q) = \frac{\partial y^i}{\partial x^j}(p)$$
$$\frac{\partial \tilde{\psi}^i}{\partial a^j}(q) = \frac{\partial y^i}{\partial a^j}(p) = 0$$

and

$$\frac{\partial \tilde{\psi}^{i+n}}{\partial x^j}(q) = \sum_k a^k \frac{\partial^2 y^i}{\partial x^j \partial x^k}(p)$$
$$\frac{\partial \tilde{\psi}^{i+n}}{\partial a^j}(q) = \frac{\partial y^i}{\partial x^j}(p)$$

So that the Jacobian matrix has the form

$$J = \begin{bmatrix} A & 0 \\ B & A \end{bmatrix}$$

where $A = \left[\frac{\partial y^i}{\partial x^j}(p)\right]$ is the Jacobian matrix of the transition function $\psi \circ \phi^{-1}$ and $B = \left[\sum_k a^k \frac{\partial^2 y^i}{\partial x^j \partial x^k}(p)\right]$. By a lemma from linear algebra, we have that $\det J = (\det A)^2$. For completeness we prove this below:

Lemma 3.2. *If a square matrix has the form*

$$M = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

where A and C are square matrices, then $\det M = \det A \det C$.

Proof. Say $A = [a^i_j]$ and $M = [m^i_j]$ and that A has size $n \times n$ and C has size $m \times m$. Then

$$\det M = \sum_{i=1}^{n+m} (-1)^{1+i} m^1_i \det M^1_i = \sum_{i=1}^{n} (-1)^{1+i} a^1_i \det M^1_i \tag{12}$$

where M_i^1 is the matrix M with the first row and i-th column removed. Then M_i^1 is of the form

$$\begin{bmatrix} A_i^1 & 0 \\ B_i & C \end{bmatrix}$$

where B_i is B with the i-th column removed. By induction on the size of A, $\det M_i^1 = \det A_i^1 \det C$—the base case is trivial. Thus we have that

$$\sum_{i=1}^{n}(-1)^{1+i}a_i^1 \det M_i^1 = \sum_{i=1}^{n}(-1)^{1+i}a_i^1 \det A_i^1 \det C = \det A \det C \tag{13}$$

By combining equations (12) and (13) we have the result. □

Exercise 12.3: Smoothness of scalar multiplication

Prove that if s is a C^∞ section of a a C^∞ vector bundle $\pi : E \to M$ and f is a C^∞ real-valued function on M then the product $fs : M \to E$ defined by

$$(fs)(p) = f(p)s(p) \in E_p, \ p \in M$$

is a C^∞ section of E.

Solution: Let $p \in M$ and let $U \subset M$ be a trivializing open set containing p with trivialization $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$. Because s is smooth, $\phi \circ s : M \to U \times \mathbb{R}^n$ is smooth so that if

$$(\phi \circ s)(p) = \left(p, \sum_i c^i(p)e_i\right)$$

then the functions $c^i : M \to \mathbb{R}$ are all smooth. Then, because ϕ is linear at each p, we have that

$$\begin{aligned}
(\phi \circ fs)(p) &= \phi(f(p)s(p)) \\
&= f(p)\phi(s(p)) && \text{(by linearity)} \\
&= f(p) \cdot \left(p, \sum_i c^i(p)e_i\right) \\
&= \left(p, \sum_i f(p)c^i(p)e_i\right)
\end{aligned}$$

Note that $p \mapsto f(p)c^i(p)$ is smooth since both f and c^i are. Thus we have that, $\phi \circ fs$ is smooth at p and since ϕ is a diffeomorphism, fs is smooth at p. Since p was arbitrary, fs is smooth.

Exercise 12.4: Coefficients relative to a smooth frame

Let $\pi : E \to M$ be a C^∞ vector bundle and $s_1, \dots, s_n$ a C^∞ frame for E over an open set U in M. Then every $e \in \pi^{-1}(U)$ can be written uniquely as a linear combination

$$e = \sum_{i=1}^{r} c^i(e)s_i(p), \quad p = \pi(e) \in U \tag{14}$$

Prove that $c^i : \pi^{-1}(U) \to \mathbb{R}$ is C^∞ for $i = 1, \dots, r$.

Solution: Let $\phi : E|_V \xrightarrow{\cong} V \times \mathbb{R}$ be a trivialization over $e \in V \subset U$ with $t_1, \dots, t_n$ its frame so that

$$e = \sum_{j=1}^{r} b^j(e) t_j(p) \tag{15}$$

for functions $b^j : E|_V \to \mathbb{R}$. Then we have that

$$\begin{aligned}
\phi(e) &= \phi\left(\sum_{j=1}^{r} b^j(e) t_j(p)\right) \\
&= \sum_{j=1}^{r} b^j(e)\phi\big(t_j(p)\big) \quad \text{(by linearity at } p\text{)} \\
&= \sum_{j=1}^{r} b^j(e)(p, e_j) \qquad \text{(by definition of the } t_j\text{)} \\
&= \left(p, \sum_{j=1}^{r} b^j(e) e_j\right)
\end{aligned}$$

which merely says that the b^j are the fiber coordinates of ϕ and thus are smooth by Proposition 6.16.

Now given the frame $s_1, \dots, s_n$ restricted to $E|_V$ we have that

$$t_j = \sum_{i=1}^{r} a_j^i s_i$$

for some functions $a_j^i : M \to \mathbb{R}$. Because the t_j are smooth and the s_i are a smooth frame, the a_j^i are smooth by Proposition 12.12. By equations (14) and (15), e is both $\sum b^j(e) t_j(p)$ and $\sum c^i(e) s_i(p)$ so that

$$\begin{aligned}
\sum_{i=1}^{r} c^i(e) s_i(p) &= \sum_{j=1}^{r} b^j(e) t_j(p) \\
&= \sum_{i,j}^{r} b^j(e) a_j^i(p) s_i(p)
\end{aligned}$$

so that by comparing coefficients we have

$$c^i(e) = \sum_{j=1}^{r} b^j(e) a_j^i(p) = \sum_{j=1}^{r} b^j(e) \cdot \left(a_j^i \circ \pi\right)(e)$$

Since we have already shown that the b^j and a_j^i are smooth and π is smooth by definition, we have that the c^i are smooth at e. Since $e \in U$ was arbitrary, the $c^i : \pi^{-1}(U) \to \mathbb{R}$ are smooth.

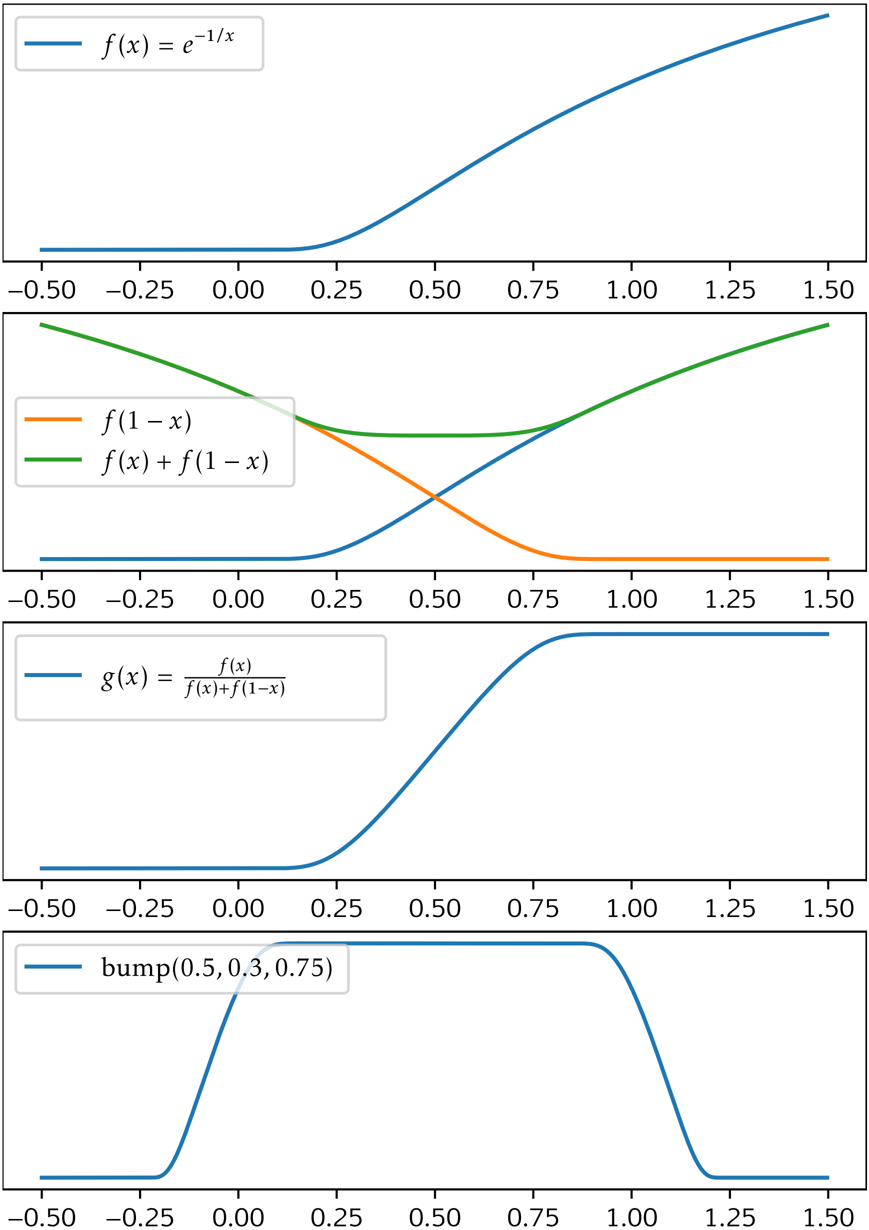

Figure 4: Construction of the C^∞ bump function.

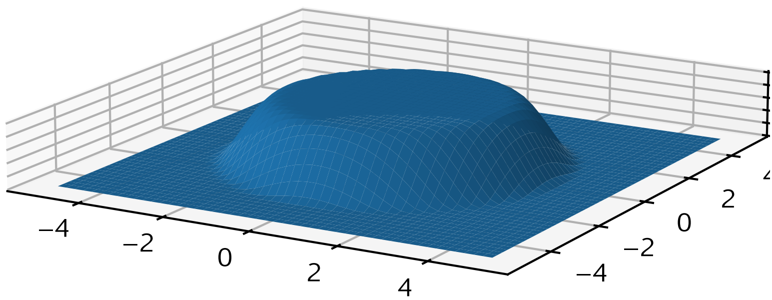

Figure 5: Bump function on $\mathbb{R}^2$ centered at 0.

A Digression on Bump Function Construction

In Figure 4, the final plot shows the bump function centered at c that is identically 1 within $]c-a, c+a[$ and supported in $[c-b, c+b]$. Its formula is

$$\operatorname{bump}(c, a, b) = x \mapsto 1 - g\left(\frac{(x-c)^2 - a^2}{b^2 - a^2}\right)$$

and in the plot we have $c = 0.5, a = 0.3, b = 0.75$. Figure 5 shows the bump function centered at 0 in $\mathbb{R}^2$ that is 1 within the disc $\|x\| < 2$ and has support $\|x\| <= 4$.

Exercise 13.1: Support of a finite sum

Prove that if $\rho_1, \ldots, \rho_m$ are real-valued functions on a manifold M, then

$$\operatorname{supp}\left(\sum \rho_i\right) \subset \bigcup \operatorname{supp} \rho_i$$

Solution: Say $x \in \operatorname{supp}(\sum \rho_i)$ and let U be a neighborhood of x. Thus U intersects a non-zero value of $\sum \rho_i$ so that at least one must be non-zero. Thus U intersects $\bigcup \rho_i^{-1}(\mathbb{R}^\times)$. Since U was arbitrary, $x \in \operatorname{cl}\left(\bigcup \rho_i^{-1}(\mathbb{R}^\times)\right)$. By Exercise A.53,

$$\operatorname{cl}\left(\bigcup \rho_i^{-1}(\mathbb{R}^\times)\right) = \bigcup \operatorname{cl}\left(\rho_i^{-1}(\mathbb{R}^\times)\right) = \bigcup \operatorname{supp} \rho_i$$

which proves the claim.

Exercise A.53:

Let A and B be subsets of a topological space S. Show that $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

Solution: Since $A \subset \mathrm{cl}(A)$ and $B \subset \mathrm{cl}(B)$ we have that $A \cup B \subset \mathrm{cl}(A) \cup \mathrm{cl}(B)$. Since $\mathrm{cl}(A) \cup \mathrm{cl}(B)$ is closed, $\mathrm{cl}(A \cup B) \subset \mathrm{cl}(A) \cup \mathrm{cl}(B)$ since the closure is the smallest closed set that contains $A \cup B$.

Conversely let $x \in \mathrm{cl}(A) \cup \mathrm{cl}(B)$ and U a neighborhood of x. If $x \in \mathrm{cl}(A)$ then U intersects A so that it intersects $A \cup B$ in which case $x \in \mathrm{cl}(A \cup B)$. Similarly if $x \in \mathrm{cl}(B)$ then also $x \in \mathrm{cl}(A \cup B)$. Thus, $\mathrm{cl}(A) \cup \mathrm{cl}(B) \subset \mathrm{cl}(A \cup B)$.

Exercise 13.2: Locally finite family and compact set

Let $\{A_\alpha\}$ be a locally finite family of subsets of a topological space S. Show that every compact set K in S has a neighborhood W that intersects only finitely many of the A_α.

Solution: Because the A_α are locally finite, each point $p \in K$ has a neighborhood U_p that intersects the A_α finitely. Since K is compact, the U_p have a finite subcover U_{p_i} for some points $p_1, \ldots, p_n$ each of which intersects finitely with some sets $A_{i,j}$, for $j = 1, \ldots, n_i$. Let $W = \bigcup_i U_{p_i}$ in which case we have

$$\begin{aligned}
W \cap \bigcup_\alpha A_\alpha &= \bigcup_i U_{p_i} \cap \bigcup_\alpha A_\alpha \\
&= \bigcup_i \left(U_{p_i} \cap \bigcup_\alpha A_\alpha \right) \\
&= \bigcup_i \left(U_{p_i} \cap \bigcup_j A_{i,j} \right) \\
&\subset \bigcup_i U_{p_i} \cap \bigcup_{i,j} A_{i,j} \\
&= W \cap \bigcup_{i,j} A_{i,j}
\end{aligned}$$

so that W intersects only with the finitely many $A_{i,j}$.

Exercise 13.3: Smooth Urysohn Lemma

(i) Let A and B be two disjoint closed sets in a manifold M. Find a C^∞ function f on M such that f is identically 1 on A and identically 0 on B.

(ii) Let A be a closed subset and U an open subset of a manifold M. Show that there is a C^∞ function f on M such that f is identically 1 on A and $\operatorname{supp} f \subset U$.

Solution:

(i) Since A and B are disjoint, their complements $M \setminus A$ and $M \setminus B$ form an open cover of M because for $x \in M$ if $x \in A$ then $x \notin B$ so that $x \in M \setminus B$.

Otherwise $x \notin A$ so that $x \in M \setminus A$. Thus $M \subset (M \setminus A) \cup (M \setminus B)$ because x was arbitrary.

Thus by Theorem 13.7, there is a C^∞ partition of unity $\rho_{M\setminus A}, \rho_{M\setminus B}$ subordinate to $\{M \setminus A, M \setminus B\}$. Let $f = \rho_{M\setminus B}$. If $x \in A$ then $x \notin M \setminus A$ so that $\rho_{M\setminus A}(x) = 0$ since $\rho_{M\setminus A}(x)$ is supported in $M \setminus A$. Then

$$\begin{aligned} 1 &= \rho_{M\setminus A}(x) + \rho_{M\setminus B}(x) \\ &= \rho_{M\setminus B}(x) \\ &= f(x) \end{aligned}$$

so that f is 1 on A. If $x \in B$ then $x \notin M \setminus B$ which contains the support of $f = \rho_{M\setminus B}$ so that $f(x) = 0$. In fact we have probed the slightly stronger claim that not only is f 0 on B but that the the support of f is contained in the complement $M \setminus B$.

(ii) $M \setminus U$ is closed and disjoint from A so that by the previous exercise there is a function f that is identically 1 on A and whose support is in $M \setminus (M \setminus U) = U$.

Exercise 13.4: Support of the pullback of a function

Let $F : N \to M$ be a C^∞ map of manifolds and $h : M \to \mathbb{R}$ a C^∞ real-valued function. Prove that $\operatorname{supp} F^*h \subset F^{-1}(\operatorname{supp} h)$.

Solution: Suppose $x \in (F^*h)^{-1}(\mathbb{R}^\times)$ so that $h(F(x)) \neq 0$. Then $F(x) \in h^{-1}(\mathbb{R}^\times) \subset \operatorname{supp} h$ so that $x \in F^{-1}(\operatorname{supp} h)$. Thus we have shown that $(F^*h)^{-1}(\mathbb{R}^\times) \subset F^{-1}(\operatorname{supp} h)$. Note that $F^{-1}(\operatorname{supp} h)$ is closed since F is continuous. Since $\operatorname{supp} F^*h$ is the smallest closed subset that contains $(F^*h)^{-1}(\mathbb{R}^\times)$, we have that $\operatorname{supp} F^*h \subset F^{-1}(\operatorname{supp} h)$, proving the claim.

Exercise 13.5: Support of the pullback by a projection

Let $f : M \to \mathbb{R}$ be a C^∞ function on a manifold M. If N is another manifold and $\pi : M \times N \to M$ is the projection onto the first factor, prove that

$$\operatorname{supp}(\pi^* f) = (\operatorname{supp} f) \times N$$

Solution: First assume that $(\pi^* f)(a, b) \neq 0$ which is to say that $f(a) \neq 0$. Thus $(a, b) \in (\operatorname{supp} f) \times N$. Since $(\operatorname{supp} f) \times N$ is closed in the product topology, and (a, b) were arbitrary non-zero values of $\pi^* f$, we have that $\operatorname{supp}(\pi^* f) \subset (\operatorname{supp} f) \times N$ since $\operatorname{supp}(\pi^* f)$ is the smallest closed set that contains the non-zero values of $\pi^* f$.

Now suppose that $(a, b) \in (\operatorname{supp} f) \times N$ and let U be an arbitrary neighborhood of (a, b). Then there exist neighborhoods U_M and U_N of a and b respectively such that $U_M \times U_N \subset U$. Since $a \in \operatorname{supp} f$, U_M must intersect with the non-zero values of f. Thus we can find an $x \in U_M$ such that $f(x) = 0$ in which case $(\pi^* f)(x, b) = f(x) = 0$. Thus $U_M \times U_N$ intersects with the non-zero values of $\pi^* f$ and so U does so as well. Since U was an arbitrary neighborhood, $(a, b) \in \operatorname{supp}(\pi^* f)$ and since (a, b) were arbitrary, $(\operatorname{supp} f) \times N \subset \operatorname{supp}(\pi^* f)$

Exercise 13.6: Pullback of a partition of unity

Suppose $\{\rho_\alpha\}$ is a partition of unity on a manifold M subordinate to an open cover $\{U_\alpha\}$ of M and $F : N \to M$ is a C^∞ map. Prove that

(i) the collection of supports $\{\operatorname{supp} F^*\rho_\alpha\}$ is locally finite;

(ii) the collection of functions $\{F^*\rho_\alpha\}$ is a partition of unity on N subordinate to the open cover $\{F^{-1}(U_\alpha)\}$.

Solution:

- Let $q \in N$. Because the $\{\operatorname{supp} \rho_\alpha\}$ are locally finite, $F(q) \in M$ has a neighborhood U with finite intersection $\{\operatorname{supp} \rho_i\} \subset \{\operatorname{supp} \rho_\alpha\}$. Thus if $x \in F^{-1}(U)$ we have that $F(x) \in U$ and thus

$$\begin{aligned} 1 &= \sum_\alpha \rho_\alpha(F(x)) \\ &= \sum_i \rho_i(F(x)) \\ &= \sum_i F^*\rho_i(x) \end{aligned}$$

so that $x \in \operatorname{supp}(\sum F^*\rho_i)$. By Lemma 13.5, $x \in \bigcup \operatorname{supp} F^*\rho_i$. Since $x \in F^{-1}(U)$ was arbitrary, $F^{-1}(U)$ intersects finitely with the $\{\operatorname{supp} F^*\rho_i\}$ of the $\{\operatorname{supp} F^*\rho_\alpha\}$. Because $q \in N$ was arbitrary, $\{\operatorname{supp} F^*\rho_\alpha\}$ is locally finite.

- For any $q \in N$ we have that $1 = \sum \rho_\alpha(F(q)) = \sum F^*\rho_\alpha(q)$ so that $\sum F^*\rho_\alpha = 1$. Thus $\{F^*\rho_\alpha\}$ is a partition of unity. By exercise 13.4, we have that $\operatorname{supp} F^*\rho_\alpha \subset F^{-1}(\operatorname{supp} \rho_\alpha) \subset F^{-1}(U_\alpha)$ since $\operatorname{supp} \rho_\alpha$ is subordinate to U_α. Thus $\operatorname{supp} F^*\rho_\alpha$ is subordinate to $F^{-1}(U_\alpha)$.

Exercise 13.7: Closure of a locally finite union

If $\{A_\alpha\}$ is a locally finite collection of subsets in a topological space, then

$$\overline{\bigcup A_\alpha} = \bigcup \overline{A_\alpha}$$

Solution: The inclusion $\bigcup \overline{A_\alpha} \subset \overline{\bigcup A_\alpha}$ is always true since if $x \in \bigcup \overline{A_\alpha}$ and U is a neighborhood of x, then since $x \in \overline{A_\alpha}$ for some α, U intersects that A_α. But then U intersects $\bigcup A_\alpha$ so that $x \in \overline{\bigcup A_\alpha}$ since U was arbitrary.

For the other direction, let $x \in \overline{\bigcup A_\alpha}$. Since $\{A_\alpha\}$ is locally finite, x has a neighborhood V that intersects finitely many $\{A_i\} \subset \{A_\alpha\}$. Now given an arbitrary neighborhood U of x, $U \cap V$ is also a neighborhood of x so that it intersects the $\bigcup A_\alpha$ since x is in the closure. Thus we have that

$$\begin{aligned} \varnothing &\neq U \cap V \cap \bigcup A_\alpha \\ &= U \cap V \cap \bigcup A_i \\ &\subset U \cap \bigcup A_i \end{aligned}$$

Thus, since $U \cap \bigcup A_i$ is non-empty, it intersects finitely with the $\{A_i\}$. Since U was arbitrary, we have that $x \in \overline{\bigcup A_i} = \bigcup \overline{A_i}$ by exercise A.53 since the $\{A_i\}$ are finite. Since each A_i is an A_α, we have that $\bigcup \overline{A_i} \subset \bigcup \overline{A_\alpha}$ so that $x \in \bigcup \overline{A_\alpha}$.

Exercise 14.1: Equality of vector fields

Show that two C^∞ vector fields X and Y on a manifold M are equal if an only if for every C^∞ function on M, we have $Xf = Yf$.

Solution: That the condition is necessary is trivial so assume that $Xf = Yf$ for any $f : M \to \mathbb{R}$. Let $p \in M$ and $(U, \phi) = (U, x^1, \dots, x^n)$ a chart around p so that $X_p = \sum a^i \frac{\partial}{\partial x^i}|_p$ and $Y_p = \sum b^i \frac{\partial}{\partial x^i}|_p$. By Proposition 13.2 we can extend each x^i to a function $\tilde{x}^i$ on all of M that agrees with x^i in a neighborhood of p. Thus we have that

$$a^i = X_p \tilde{x}^i = Y_p \tilde{x}^i = b^i$$

so that $X_p = Y_p$. Since p was arbitrary, $X = Y$.

Exercise 14.2: Vector field on an odd sphere

Let $x^1, y^1, \dots, x^n, y^n$ be the standard coordinates on $\mathbb{R}^{2n}$. The unit sphere S^{2n-1} in $\mathbb{R}^{2n}$ is defined by the equation $\sum_{i=1}^n \left(x^i\right)^2 + \left(y^i\right)^2 = 1$. Show that

$$X = \sum_{i=1}^n -y^i \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial y^i}$$

is a nowhere-vanishing smooth vector field on S^{2n-1}.

Solution: We have that $\sum -y^i x^i + x^i y^i = 0$ so that this is a tangent vector at the point $\left(x^i, y^i\right)$ by exercise 11.1. To find the integral curves of this flow, consider a point $\left(a^i, b^i\right) \in S^{2n-1}$ and let

$$c^i = \sqrt{(a^i)^2 + (b^i)^2}$$

Because $\left(a^i\right)^2 \le \left(a^i\right)^2 + \left(b^i\right)^2 = \left(c^i\right)^2$, we have that $-c^i \le a^i \le c^i$. Because $c^i \cos t$ ranges from $-c^i$ to c^i there must exist a t_0^i such that $c^i \cos t_0^i = a^i$ by the intermediate value theorem. In that case we have that

$$\begin{aligned}
\left(b^i\right)^2 &= \left(a^i\right)^2 + \left(b^i\right)^2 - \left(a^i\right)^2 \\
&= \left(c^i\right)^2 - \left(c^i \cos t_0^i\right)^2 \\
&= \left(c^i\right)^2 \left(1 - \cos^2 t_0^i\right) \\
&= \left(c^i\right)^2 \sin^2 t_0^i
\end{aligned}$$

so that $b^i = c^i \sin t_0^i$. In that case the curve

$$c(t) = \left(c^i \cos(t - t_0^i), c^i \sin(t - t_0^i)\right)$$

satisfies

$$c(0) = \left(c^i \cos t_0^i, c^i \sin t_0^i\right) = \left(a^i, b^i\right)$$

and

$$\begin{aligned} c'(t) &= \sum_{i=1}^{n} -c^i \sin(t - t_0^i)\frac{\partial}{\partial x^i} + c^i \cos(t - t_0^i)\frac{\partial}{\partial y^i} \\ &= \sum_{i=1}^{n} -y^i(c(t))\frac{\partial}{\partial x^i} + x^i(c(t))\frac{\partial}{\partial y^i} \end{aligned}$$

and so is the unique flow at $\left(a^i, b^i\right)$. Each integral curve thus corresponds uniquely to choosing the set of c^i which satisfy

$$\sum_{i=1}^{n} \left(c^i\right)^2 = \sum_{i=1}^{n} \left(a^i\right)^2 + \left(b^i\right)^2 = 1$$

so that the space of curves is itself a sphere with dimension $n-1$.

Figure 6 shows this vector field and its integral curves for S^3 in the chart $\{(x, y, z, w) \mid w > 0\}$. The circle going around the "equator" of this chart is the curve $(\cos t, \sin t, 0, 0)$. The line going straight down the z-axis is the curve $(0, 0, \cos t, \sin t)$.

Exercise 14.3: Maximal integral curve on a punctured line

Let M be $\mathbb{R} - \{0\}$ and let X be the vector field d/dx on M. Find the maximal integral curve of X starting at $x = 1$.

Solution: The curve is $c(x) = x + 1$ defined on $]-1, \infty[$ in which case $c'(x) = d/dx$.

Exercise 14.4: Integral curves in the plane

Find the integral curves of the vector field

$$X_{(x,y)} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \quad \text{on } \mathbb{R}^2$$

Solution: The integral curve at the point $(a, b) \in \mathbb{R}^2$ is $c(t) = \left(ae^t, be^{-t}\right)$ because then we have

$$\begin{aligned} c(0) &= (a, b) \\ c'(t) &= ae^t\frac{\partial}{\partial x} - be^{-t}\frac{\partial}{\partial y} \end{aligned}$$

The graph of various integral curves of this vector field is given in Figure 7.

Exercise 14.5: Maximal integral curve in the plane

Find the maximal integral curve $c(t)$ starting at the point $(a, b) \in \mathbb{R}^2$ of the vector field $X_{(x,y)} = \partial/\partial x + x\partial/\partial y$ on $\mathbb{R}^2$.

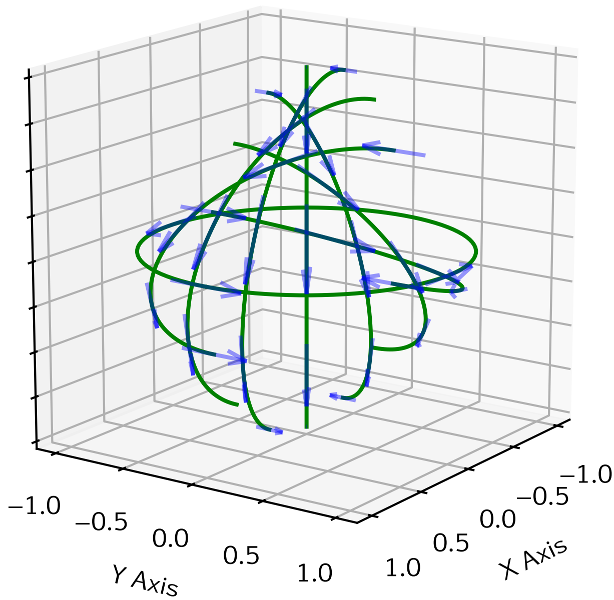

Figure 6: Problem 14.2

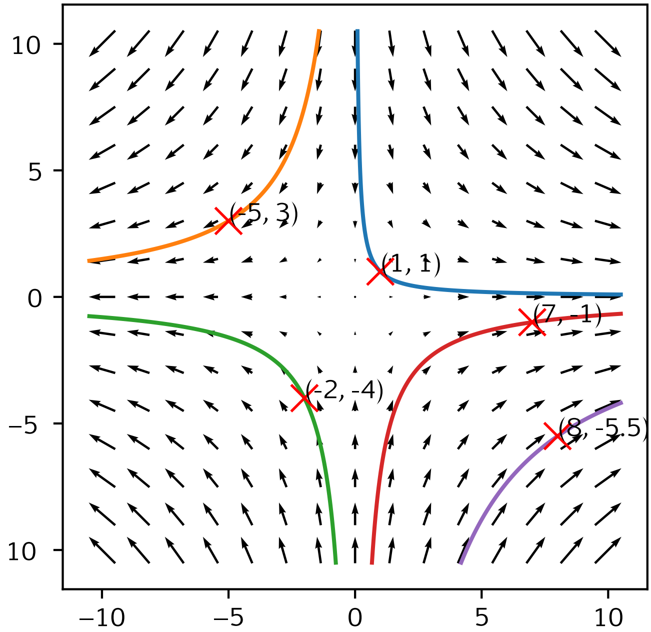

Figure 7: Problem 14.4

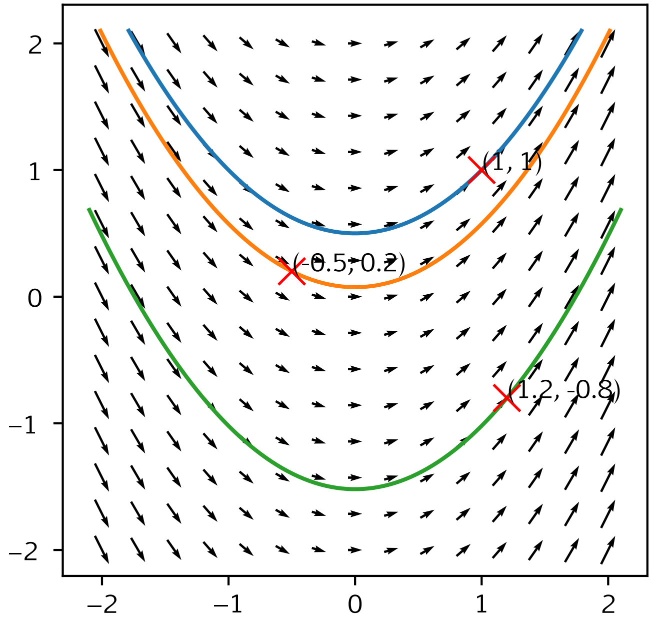

Figure 8: Problem 14.5

Solution: The curve is $c(t) = \left(t + a, \frac{1}{2}t^2 + ta + b\right)$ because then

$$c'(t) = \frac{\partial}{\partial x} + (t + a)\frac{\partial}{\partial y}$$

This is the hamiltonian phase space for a constant acceleration of 1. The x coordinate is velocity and y is position. The initial point a specifies the starting velocity and b specifies the starting position. Various integral curves and the vector field are graphed in Figure 8.

Exercise 14.7: Maximal integral curve

Let X be the vector field $x d/dx$ on R. For each p in $\mathbb{R}$, find the maximal integral curve $c(t)$ of X starting at p.

Solution: This is $c(t) = pe^t$ because then $c(0) = pe^0 = p$ and $c'(t) = pe^t d/dx$.

Exercise 14.8: Maximal integral curve

Let X be the vector field $x^2 d/dx$ on the real line R. For each $p > 0$ in R, find the maximal integral curve of X with initial point p.

Solution: Following the example in the text, we have that

$$x(t) = -\frac{1}{t + C}$$

The condition $x(0) = p$ means that $p = -1/C$ so that $C = -1/p$. Thus we have the equation

$$x(t) = -\frac{1}{t - \frac{1}{p}}$$

which is defined as long as $t \neq \frac{1}{p}$. Thus the maximal interval containing 0 is $\left]\infty, \frac{1}{p}\right[$.

N.B. in the exercises that follow, we identify a vector field X with a function $X : C^\infty(M) \to C^\infty(M)$ that is a derivation on products even though this correspondence will not be proved until section 19. This allows to avoid awkward calculations with X_p and calculate globally using the vector fields operation on functions.

Exercise 14.10: Lie bracket of vector fields

If f and g are C^∞ functions and X and Y are C^∞ vector fields on a manifold M, show that

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$$

Solution: Apply $[fX, gY]$ to a smooth function h and use the fact that X and Y are derivations on products of functions:

$$\begin{aligned}
[f \cdot X, g \cdot Y]\,h &= f \cdot X(g \cdot Yh) - g \cdot Y(f \cdot Xh) \\
&= f \cdot Xg \cdot Yh + f \cdot g \cdot X(Yh) - g \cdot Yf \cdot Xh - g \cdot f \cdot Y(Xh) \\
&= f \cdot g \cdot (X(Yh) - Y(Xh)) + f \cdot Xg \cdot Yh - g \cdot Yf \cdot Xh \\
&= f \cdot g \cdot [X,Y]\,h + f \cdot Xg \cdot Yh - g \cdot Yf \cdot Xh \\
&= \big(f \cdot g \cdot [X,Y] + f \cdot Xg \cdot Y - g \cdot Yf \cdot X\big)h
\end{aligned}$$

so that $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$

Exercise 14.11: Lie bracket of vector fields on $\mathbb{R}^2$

Compute the Lie bracket

$$\left[-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right]$$

on $\mathbb{R}^2$.

Solution: If

$$\left[-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right] = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$$

then

$$
\begin{aligned}
a &= \left[-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right] x \\
&= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\frac{\partial x}{\partial x} - \frac{\partial}{\partial x}\left(-y\frac{\partial x}{\partial x} + x\frac{\partial x}{\partial y}\right) \\
&= -y\frac{\partial 1}{\partial x} + x\frac{\partial 1}{\partial y} + \frac{\partial y}{\partial x} \\
&= 0 + 0 + 0 = 0
\end{aligned}
$$

and

$$
\begin{aligned}
b &= \left[-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right] y \\
&= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\frac{\partial y}{\partial x} - \frac{\partial}{\partial x}\left(-y\frac{\partial y}{\partial x} + x\frac{\partial y}{\partial y}\right) \\
&= -\frac{\partial}{\partial x}x \\
&= -1
\end{aligned}
$$

so that

$$\left[-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right] = -\frac{\partial}{\partial y}$$

Exercise 14.12: Lie bracket in local coordinates

Consider two C^∞ vector fields X, Y on $\mathbb{R}^n$:

$$X = \sum a^i \frac{\partial}{\partial x^i} \quad Y = \sum b^i \frac{\partial}{\partial x^i}$$

where a^i, b^j are C^∞ functions on $\mathbb{R}^n$. Since $[X, Y]$ is also a C^∞ vector field on $\mathbb{R}^n$,

$$[X, Y] = \sum c^i \frac{\partial}{\partial x^i}$$

for some C^∞ functions c^i. Find the formula for c^i in terms of a^i and b^i.

Solution:

$$\begin{aligned}
c^i &= [X,Y]x^i \\
&= \left(\sum_j a^j \frac{\partial}{\partial x^j}\right)\left(\sum_j b^j \frac{\partial x^i}{\partial x^j}\right) - \left(\sum_j b^j \frac{\partial}{\partial x^j}\right)\left(\sum_j a^j \frac{\partial x^i}{\partial x^j}\right) \\
&= \left(\sum_j a^j \frac{\partial}{\partial x^j}\right) b^i - \left(\sum_j b^j \frac{\partial}{\partial x^j}\right) a^i \\
&= \sum_j a^j \frac{\partial b^i}{\partial x^j} - b^j \frac{\partial a^i}{\partial x^j}
\end{aligned}$$

Exercise 14.13: Vector field under a diffeomorphism

Let $F : N \to M$ be a C^∞ diffeomorphism of manifolds. Prove that if g is a C^∞ function and X a C^∞ vector field on N, then

$$F_*(gX) = (g \circ F^{-1})F_*X$$

Solution: Given an arbitrary C^∞ function f on M, we have by definition that $F_*(gX)f = gX(f \circ F)$. Applying this function to an arbitrary point $p \in N$, let $q = F(p)$ in which case we have that

$$\begin{aligned}
(F_*(gX)f)(p) &= g(p)X_p(f \circ F) \\
&= g\left(F^{-1}(q)\right) F_{*,p}(X_p)(f) \\
&= g\left(F^{-1}(q)\right) F_*(X)_q(f) \\
&= \left(\left(g \circ F^{-1}\right) F_*(X)f\right)(q)
\end{aligned}$$

so that $F_*(gX)f = \left(g \circ F^{-1}\right) F_*(X)f$. Since f was arbitrary, we have that $F_*(gX) = \left(g \circ F^{-1}\right) F_*(X)$ by Exercise 14.1.

Exercise 14.14: Lie bracket under a diffeomorphism

Let $F : N \to M$ be a C^∞ diffeomorphism of manifolds. Prove that if X Y are C^∞ vector fields on N, then

$$F_*[X,Y] = [F_*X, F_*Y]$$

Solution: The vector fields X and F_*X are F-related and similarly for Y and F_*Y, in which case $[X,Y]$ and $[F_*X, F_*Y]$ are F-related by Proposition 14.17. Thus $F_*[X,Y] = [F_*X, F_*Y]$ by the definition of F-relatedness.

Chapter 4

Exercise 15.1: Matrix exponential

For $X \in \mathbb{R}^{n\times n}$, define the partial sum $s_m = \sum_{k=0}^{m} X^k/k!$.

(i) Show that for $l \geq m$,

$$\|s_l - s_m\| \leq \sum_{k=m+1}^{l} \frac{\|X\|^k}{k!}$$

(ii) Conclude that s_m is a Cauchy sequence in $\mathbb{R}^{n\times n}$ and therefore converges to a matrix which we denote by e^X.

Solution:

- $$\begin{aligned}\|s_l - s_m\| &= \left\| \sum_{k=0}^{l} \frac{X^k}{k!} - \sum_{k=0}^{m} \frac{X^k}{k!} \right\| \\ &= \left\| \sum_{k=m+1}^{l} \frac{X^k}{k!} + \sum_{k=0}^{m} \frac{X^k}{k!} - \sum_{k=0}^{m} \frac{X^k}{k!} \right\| \\ &= \left\| \sum_{k=m+1}^{l} \frac{X^k}{k!} \right\| \\ &\leq \sum_{k=m+1}^{l} \left\| \frac{X^k}{k!} \right\|\end{aligned}$$

- Since
$$\sum_{k=0}^{\infty} \frac{\|X\|^k}{k!} = e^{\|X\|}$$
we have that for any ϵ, there is an N such that for any $m \geq n \geq N$
$$\sum_{k=m+1}^{l} \frac{\|X\|^k}{k!} < \epsilon$$
But this is simply $\|s_l - s_m\|$ so that s_m is a Cauchy sequence.

Exercise 15.2: Product rule for matrix-valued functions

Let $]a,b[$ be an open interval in $\mathbb{R}$. Suppose $A :]a,b[\to \mathbb{R}^{m\times n}$ and $B :]a,b[\to \mathbb{R}^{n\times p}$ are $m \times n$ and $n \times p$ matrices respectively whose entries are differentiable functions of $t \in]a,b[$. Prove that for $t \in]a,b[$,

$$\frac{d}{dt} A(t)B(t) = A'(t)B(t) + A(t)B'(t)$$

where $A'(t) = (dA/dt)(t)$ and $B'(t) = (dB/dt)(t)$.

Solution: Say $A = [a^i_j]$ and $B = [b^i_j]$ where a^i_j and b^i_j are functions of $t \in \,]a,b[$. Then

$$\begin{aligned}
\frac{d}{dt}A\cdot B &= \left[\frac{d}{dt}\sum_k a^i_k b^k_j\right] && \text{(by matrix multiplication)}\\
&= \left[\sum_k \frac{da^i_k}{dt}b^k_j + a^i_k\frac{db^k_j}{dt}\right] && \text{(by the product rule)}\\
&= \left[\sum_k \frac{da^i_k}{dt}b^k_j\right] + \left[\sum_k a^i_k\frac{db^k_j}{dt}\right]\\
&= A'(t)B(t) + A(t)B'(t) && \text{(by matrix multiplication)}
\end{aligned}$$

Exercise 15.5: Differential of the multiplication map

Let G be a Lie group with multiplication map $\mu : G\times G \to G$, and let $l_a : G \to G)$ and $r_b : G \to G)$ be left and right multiplication by a and $b \in G$, respectively. Show that the differential of μ at $(a,b)\in G\times G$ is

$$\mu_{*,(a,b)}(X_a, Y_b) = (r_b)_*X_a + (l_a)_*Y_b$$

for $X_a \in T_a(G)$ and $Y_b \in T_b(G)$.

Solution: Let $c_a(t) : \mathbb{R} \to G$ and $c_b(t) : \mathbb{R} \to G$ be curves at a and b respectively such that $c_a'(0) = X_a$ and $c_b'(0) = Y_b$. Then $\overline{c_a}(t) = (c_a(t), b)$ and $\overline{c_b}(t) = (a, c_b(t))$ are both curves at (a,b) such that, by Exercise 8.7,

$$\overline{c_a}'(0) = (c_a'(0), 0) = (X_a, 0) \tag{16}$$

since b is constant and similarly $\overline{c_b}'(t) = (0, c_b'(t)) = (0, Y_b)$. We also have that

$$(\mu\circ\overline{c_a})(t) = \mu(c_a(t), b) = r_b(c_a(t)) = (r_b\circ c_a)(t) \tag{17}$$

and similarly $\mu\circ\overline{c_b} = l_a\circ c_b$. Thus we can calculate

$$\begin{aligned}
\mu_{*,(a,b)}(X_a, 0) &= \mu_{*,(a,b)}(\overline{c_a}'(0)) && \text{(by (16))}\\
&= (\mu_{*,(a,b)}\circ(\overline{c_a})_*)\left(\frac{d}{dt}\bigg|_0\right)\\
&= (r_b)_*\circ(c_a)_*\left(\frac{d}{dt}\bigg|_0\right) && \text{(by (17) and functoriality)}\\
&= (r_b)_*(c_a'(0))\\
&= (r_b)_*(X_a)
\end{aligned}$$

and similiarly $\mu_{*,(a,b)}(0, Y_b) = (l_a)_*(Y_b)$. Thus finally we have

$$\mu_{*,(a,b)}(X_a, Y_b) = \mu_{*,(a,b)}(X_a, 0) + \mu_{*,(a,b)}(0, Y_b) = (r_b)_*X_a + (l_a)_*Y_b$$

Exercise 15.6: Differential of the inverse map

Let G be a Lie group with multiplication map $\mu : G \times G \to G$, inverse map $\iota : G \to G$ and identity element e. Show that the differential of the inverse map at $a \in G$,

$$i_{*,a} : T_a G \to T_{a^{-1}} G)$$

is given by

$$i_{*,a}(Y_a) = -(r_{a^{-1}})_{*,e}(l_{a^{-1}})_{*,a} Y_a$$

Solution: Let $c(t) : \mathbb{R} \to G$ be a curve at $a \in G$ such that $c'(0) = Y_a$ and define the curve $\bar{c} : R \to G \times G$ as $\bar{c}(t) = (c(t), (\iota \circ c)(t))$. Then

$$\bar{c}'(0) = (c'(0), \iota_{*,a}(c'(0))) = (Y_a, i_{*,a} Y_a)$$

and since $\mu \circ \bar{c}(t) = \mu(c(t), \iota(c(t))) = e$ is constant, we have

$$\begin{aligned}
0 &= (\mu \circ \bar{c})'(0) \\
&= \mu_{*,(a,a^{-1})} \circ \bar{c}_* \left(\frac{d}{dt}\bigg|_0 \right) \\
&= \mu_{*,(a,a^{-1})}(\bar{c}'(0)) \\
&= \mu_{*,(a,a^{-1})}(Y_a, \iota_{*,a} Y_a) \\
&= (r_{a^{-1}})_* Y_a + (l_a)_* i_{*,a} Y_a \quad \text{(by Exercise 15.5)}
\end{aligned}$$

so that

$$(l_a)_* i_{*,a} Y_a = -(r_{a^{-1}}) Y_a$$

and therefore

$$\begin{aligned}
i_{*,a} Y_a &= -(l_a)_*^{-1}(r_{a^{-1}}) Y_a \\
&= -(l_a^{-1})_*(r_{a^{-1}}) Y_a \quad \text{(by functoriality)} \\
&= -(l_{a^{-1}})_*(r_{a^{-1}})_* Y_a \quad \text{(definition of left multiplication)}
\end{aligned}$$

On the other hand, by using the curve $\bar{c}(t) = ((\iota \circ c)(t), c(t))$ we get the equation

$$0 = \mu_{*,(a^{-1},a)}(\iota_{*,a} Y_a, Y_a) = (r_a)_* \iota_{*,a} Y_a + (l_{a^{-1}})_* Y_a$$

which yields

$$i_{*,a} Y_a = -(r_{a^{-1}})_*(l_{a^{-1}})_* Y_a$$

Exercise 15.7: Differential of the determinant map at A

Show that the differential of the determinant map $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ at $A \in \mathrm{GL}(n, \mathbb{R})$ is given by

$$(\det)_{*,A}(AX) = \det A \cdot \operatorname{tr} X \quad \text{for } X \in \mathbb{R}^{n \times n}$$

Solution: First just a note that the general formula is

$$(\det)_{*,A}(X) = \det A \operatorname{tr}(A^{-1} X)$$

Consider the curve $c(t) = Ae^{tX}$ which satisfies $c(0) = A$ and $c'(0) = AX$ since $c'(t) = AXe^{tX}$. Then

$$\begin{aligned}
(\det)_{*,A}(AX) &= (\det)_{*,A}\big(c'(0)\big) \\
&= \frac{d}{dt}\bigg|_0 \det(c(t)) && \text{(Proposition 8.18)} \\
&= \frac{d}{dt}\bigg|_0 \det(Ae^{tX}) \\
&= \frac{d}{dt}\bigg|_0 \det(A)\det(e^{tX}) \\
&= \det(A)\,\frac{d}{dt}\bigg|_0 \det(e^{tX}) \\
&= \det(A)\,\frac{d}{dt}\bigg|_0 e^{t\,\mathrm{tr}\,X} && \text{(Proposition 15.20)} \\
&= \det(A)\,\mathrm{tr}\,X\, e^{t\,\mathrm{tr}\,X}\Big|_0 \\
&= \det(A)\,\mathrm{tr}\,X
\end{aligned}$$

Exercise 15.8: Special linear group

Use Exercise 15.7 to show that 1 is a regular value of the determinant map. This gives a quick proof that the special linear group $\mathrm{SL}(n, \mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$.

Solution: We must show that if $\det A = 1$ then $(\det)_{*,A} : T_A \mathrm{GL}(n, \mathbb{R}) \to T_1\mathbb{R}$ is surjective on $T_1\mathbb{R} = \mathbb{R}$. From the previous exercise, let

$$X = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

which is to say, 1 in the first row and column and 0 otherwise. Then

$$(\det)_{*,A}(AX) = \det A \cdot \mathrm{tr}\,X = 1 \cdot 1 = 1$$

so that $(\det)_{*,A}$ is surjective.

A Digression on the Exterior Derivative

I did not feel totally convinced by the uniqueness proof in Tu's book that the exterior derivative definition does not depend on the choice of chart, so I decided to calculate it explicitly. I feel more convinced of his proof now, but I still thought it was nifty to see how the explicit calculation works. So first some definitions. Given a manifold M, a $k-1$-form $\omega \in \Omega^k(M)$ and a chart $(U, \phi = x^1, \dots, x^n)$, define the exterior derivative

$$d_U\omega = \sum_i \frac{\partial f}{\partial x^i} dx^i \wedge dx^I$$

where $\omega = f dx^I$ in the chart U.

Then we already have that $dd\omega = 0$ by Proposition 4.7.ii. Next we show the following:

Lemma 4.1. *Given a manifold M and charts $(U, \phi = x^1, \dots, x^n)$ and $(V, \psi = y^1, \dots, y^n)$, the k-form*

$$\sum_J \frac{\partial y^I}{\partial x^J} dx^J = d\omega$$

for a $k-1$-form ω, where J ranges over all multi-indices of size k and I is a multi-index of size k.

Proof. This is proved by induction and the base case $k = 1$ is trivial

$$\sum_j \frac{\partial y^i}{\partial x^j} = dy^i$$

Now we calculate

$$\begin{aligned}
&d \sum_{|J|=k-1} y^{i_1} \frac{\partial y^{I \setminus i_1}}{\partial x^J} dx^J \\
&\qquad = \sum_{\substack{|J|=k-1 \\ j=1,\dots,k}} \frac{\partial y^{i_1}}{\partial x^j} \cdot \frac{\partial y^{I \setminus i_1}}{\partial x^J} dx^j \wedge dx^J + y^{i_1} d\left(\sum_{|J|=k-1} \frac{\partial y^{I \setminus i_1}}{\partial x^J} dx^J \right) \\
&\qquad = \sum_{\substack{|J|=k \\ j_i \in J}} (-1)^{i+1} \frac{\partial y^{i_1}}{\partial x^{j_i}} \cdot \frac{\partial y^{I \setminus i_1}}{\partial x^{J \setminus j_i}} dx^J + y^{i_1} dd\omega \quad \text{(by induction)} \\
&\qquad = \sum_{|J|=k} \frac{\partial y^I}{\partial x^J} dx^J \quad \text{(by the determinant formula)}
\end{aligned}$$

so that

$$\omega = \sum_{|J|=k-1} y^{i_1} \frac{\partial y^{I \setminus i_1}}{\partial x^J} dx^J$$

is the desired $k-1$-form. □

Now we can prove chart invariance of the exterior derivative.

Theorem 4.2. *Let M be a manifold and $(U, \phi = x^1, \dots, x^n)$ and $(V, \psi = y^1, \dots, y^n)$ charts on M. If $\omega \in \Omega^k(M)$ is a k-form on M then $d_U\omega = d_V\omega$ on $U \cap V$.*

Proof. Let $\omega = \sum_J a_J dx^J = \sum_I b_I dy^I$. Then we have that

$$b_I = \sum_J a_J \frac{\partial x^J}{\partial y^I} \tag{18}$$

$$dy^I = \sum_J \frac{\partial y^I}{\partial x^J} dx^J \tag{19}$$

Then

$$\begin{aligned}
d_V \sum_I b_I dy^I &= \sum_{I,i} \frac{\partial b_I}{\partial y^i} dy^i \wedge dy^I \\
&= \sum_{I,i} \frac{\partial}{\partial y^i} \left(\sum_J a_J \frac{\partial x^J}{\partial y^I} \right) dy^i \wedge dy^I && \text{(by (18))} \\
&= \sum_{I,i,J} \frac{\partial a_J}{\partial y^i} \frac{\partial x^J}{\partial y^I} dy^i \wedge dy^I + \sum_{I,i,J} a_J \frac{\partial}{\partial y^i} \frac{\partial x^J}{\partial y^I} dy^i \wedge dy^I
\end{aligned}$$

Now consider the second term

$$\begin{aligned}
\sum_{I,i,J} a_J \frac{\partial}{\partial y^i} \frac{\partial x^J}{\partial y^I} dy^i \wedge dy^I &= \sum_J a_J d_V \sum_I \frac{\partial x^J}{\partial y^I} dy^I \\
&= \sum_J a_J d_V d_V \omega && \text{(by Lemma 4.1)} \\
&= 0
\end{aligned}$$

and the first term

$$\begin{aligned}
\sum_{I,i,J} \frac{\partial a_J}{\partial y^i} \frac{\partial x^J}{\partial y^I} dy^i \wedge dy^I &= \sum_{\substack{I,i \\ J,j}} \frac{\partial a_J}{\partial y^i} \frac{\partial y^i}{\partial x^j} \frac{\partial x^J}{\partial y^I} dx^j \wedge dy^I && \text{(by (19))} \\
&= \sum_{I,J,j} \frac{\partial a_J}{\partial x^j} \frac{\partial x^J}{\partial y^I} dx^j \wedge dy^I && \text{(the chain rule)} \\
&= \sum_{\substack{I,J \\ K,j}} \frac{\partial a_J}{\partial x^j} \frac{\partial x^J}{\partial y^I} \frac{\partial y^I}{\partial x^K} dx^j \wedge dx^K && \text{(by (19))}
\end{aligned}$$

(By the functoriality of the determinant and the chain rule:)

$$\begin{aligned}
&= \sum_{J,K,j} \frac{\partial a_J}{\partial x^j} \frac{\partial x^J}{\partial x^K} dx^j \wedge dx^K \\
&= \sum_{J,j} \frac{\partial a_J}{\partial x^j} dx^j \wedge dx^J && \left(\text{since } \frac{\partial x^J}{\partial x^K} = \delta^J_K\right) \\
&= d_U \sum_J a_J dx^J && \text{(by definition)}
\end{aligned}$$

so that

$$d_V \sum_I b_I dy^I = d_U \sum_J a_J dx^J$$

as desired. □