Chapter 1

Exercise p. 12: Determine with what velocity a stone must be thrown in order that it fly infinitely far from the surface of the Earth.

Solution: Assume that $x(t) : \mathbb{R} \to \mathbb{R}$ is the path of a thrown stone. Thus we have that

$$\ddot{x} = -\frac{gr_0^2}{(r_0 + x)^2} \tag{1}$$

where $g = 9.8\frac{\text{m}}{\text{s}^2}$ and r_0 is the radius of the Earth, approximately 6,371 km. We also assume the stone begins at the surface of the Earth, so that $x(0) = 0$ and that $x(t)$ increases unboundedly, which is to say that the stone escapes Earth's gravitation.

The chain rule gives us the following

$$\frac{d}{dt}\frac{\dot{x}^2}{2} = \dot{x}\ddot{x} \tag{2}$$

We proceed by integrating both sides of (2) over t.

By the fundamental theorem of calculus, the left hand side gives

$$\int_0^{t_0} \frac{d}{dt}\frac{\dot{x}^2}{2} dt = \frac{\dot{x}(t_0)^2}{2} - \frac{\dot{x}(0)^2}{2}$$

By substituting (1) for $\ddot{x}$ and using the change of variables formula, the right hand side gives

$$\begin{aligned}
\int_0^{t_0} \dot{x}\ddot{x} dt &= -gr_0^2 \int_0^{t_0} \frac{\dot{x}}{(r_0 + x)^2} dt \\
&= -gr_0^2 \int_0^{x(t_0)} \frac{1}{(r_0 + x)^2} dx \\
&= gr_0^2 \left. \frac{1}{r_0 + x} \right|_0^{x(t_0)} \\
&= \frac{gr_0^2}{r_0 + x(t_0)} - gr_0
\end{aligned}$$

Combining both sides and solving for $\dot{x}(0)$ gives

$$\begin{aligned}
\dot{x}(0) &= \sqrt{2\left(gr_0 - \frac{gr_0^2}{r_0 + x(t_0)} + \frac{\dot{x}(t_0)^2}{2}\right)} \\
&\geq \sqrt{2gr_0 - \frac{2gr_0^2}{r_0 + x(t_0)}} \\
&= \sqrt{2gr_0 - \epsilon}
\end{aligned}$$

where $\epsilon = \frac{2gr_0^2}{r_0+x(t_0)}$. Since we have assumed that x increases unboundedly, escaping Earth, we can make ϵ arbitrarily small by increasing t_0. Since the above holds for all ϵ, we have that the initial velocity

$$\begin{aligned} \dot{x}(0) &\geq \sqrt{2gr_0} \\ &= \sqrt{2 \cdot .0098 \frac{\text{km}}{\text{s}^2} \cdot 6371\text{km}} \\ &= \sqrt{124.87 \frac{\text{km}^2}{\text{s}^2}} \\ &= 11.17 \frac{\text{km}}{\text{s}} \end{aligned}$$

Another way to show this is that in Hamiltonian phase space, any solution curve must lie in the level set curve

$$\frac{y^2}{2} + \frac{-gr_0^2}{r_0 + x} = E$$

where $U(x) = \frac{-gr_0^2}{r_0+x}$ is the potential and E is an energy level. Thus we have that

$$E = \frac{\dot{x}(t)^2}{2} + \frac{-gr_0^2}{r_0 + x(t)} \geq \frac{-gr_0^2}{r_0 + x(t)}$$

Since we have assumed we can make $x(t)$ arbitrarily large, we can make $\frac{-gr_0^2}{r_0+x(t)}$ arbitrarily close to 0 so that $E \geq 0$.

Thus we have that

$$\begin{aligned} \dot{x}(0) &= \sqrt{2\left(E + \frac{gr_0^2}{r_0 + x(0)}\right)} \\ &= \sqrt{2E + 2gr_0} \\ &\geq \sqrt{2gr_0} \end{aligned}$$

which also demonstrates the result.

Exercise p. 18-4: Show that the time it takes to go from x_1 to x_2 (in one direction) is equal to

$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{\sqrt{2\big(E - U(x)\big)}}$$

Solution: Given a curve $c(t) = \big(x(t), y(t)\big)$ such that $\dot{x}(t) = y(t)$, we have that for $y(t) \neq 0$

$$\begin{aligned} t_2 - t_1 &= \int_{t_1}^{t_2} dt \\ &= \int_{t_1}^{t_2} \frac{y(t)}{y(t)} dt \\ &= \int_{t_1}^{t_2} \frac{\dot{x}(t)}{\sqrt{2(E - U(x(t)))}} dt \\ &= \int_{x_1}^{x_2} \frac{dx}{\sqrt{2(E - U(x))}} \end{aligned}$$

by the change of variables formula for integrals.

The only question is whether this integral extends to the limit where $y(t_0) = 0$ and the potential is at its highest value $U(x_0) = E$ for $x_0 = x(t_0)$. We will show we can extend the integral to these points when $U'(x_0) \neq 0$.

Thus assume that $U'(x_0) \neq 0$ and in particular that $U'(x_0) < 0$. Because U' is continuous, we can find a neighborhood V of x_0 within which $U(x) < 0$ for all $x \in V$. Thus in that neighborhood $U(x)$ is monotonically decreasing. Choose a point $x_1 > x_0$ such that $x_1 \in V$ and form the sequence

$$y_0 = x_1 - x_0, y_n = \frac{y_0}{2^n}$$

Then we have the following facts:

$$\begin{aligned} y_{n+1} &= \frac{y_0}{2 \cdot 2^{n-1}} = \frac{y_n}{2} \\ y_n - y_{n+1} &= y_n - \frac{y_n}{2} = \frac{y_n}{2} = y_{n+1} \\ \frac{y_{n+1}}{y_n} &= \frac{y_n}{2y_n} = \frac{1}{2} \\ \lim_{n \to \infty} y_n &= 0 \end{aligned}$$

Now consider the improper integral

$$\lim_{x \to x_0} \int_{x}^{x_1} \frac{dx}{\sqrt{2(E - U(x))}}$$

For any x, choose an n so that $y_n < x - x_0$ in which case $[y_{i+1} + x_0, y_i + x_0]$ for $i = 0, \ldots, n-1$ is a partition of $[x, x_1]$. Since U is monotonically decreasing in V, $\sqrt{2(E - U(x))}$ is monotonically increasing, so that $\frac{1}{\sqrt{2(E-U(x))}}$ is monotonically decreasing. Thus the supremum of the function $\frac{1}{\sqrt{2(E-U(x))}}$ in each interval $[y_{i+1} + x_0, y_i + x_0]$ occurs at $y_{i+1} + x_0$. By Taylor's theorem, write $U(x) = E + U'(x_0)(x - x_0) + r(x)$ where $\lim_{x\to x_0} \frac{r(x)}{x-x_0} = 0$. Equivalently, $\lim_{x\to 0} \frac{r(x+x_0)}{x} = 0$. Then we can bound the integral by the upper partial Riemann sum as follows:

$$\begin{aligned}
\int_x^{x_1} \frac{dx}{\sqrt{2(E-U(x))}} &\leq \sum_{i=1}^{n} \frac{(y_i + x_0) - (y_{i+1} + x_0)}{\sqrt{2\big(E - U(y_{i+1} + x_0)\big)}} \\
&= \sum_{i=1}^{n} \frac{y_i - y_{i+1}}{\sqrt{2\big(E - E + U'(x_0)(y_{i+1} + x_0 - x_0) + r(y_{i+1} + x_0)\big)}} \\
&= \sum_{i=1}^{n} \frac{y_{i+1}}{\sqrt{-2U'(x_0)(y_{i+1}) - 2r(y_{i+1} + x_0)}} \\
&= \sum_{i=1}^{n} \frac{y_{i+1}}{\sqrt{y_{i+1}} \cdot \sqrt{-2U'(x_0) - 2\frac{r(y_{i+1}+x_0)}{y_{i+1}}}} \\
&= \sum_{i=1}^{n} \frac{\sqrt{y_{i+1}}}{\sqrt{-2U'(x_0) - 2\frac{r(y_{i+1}+x_0)}{y_{i+1}}}}
\end{aligned}$$

so that

$$\lim_{x\to 0} \int_x^{x_1} \frac{dx}{\sqrt{2(E-U(x))}} \leq \sum_{i=1}^{\infty} \frac{\sqrt{y_{i+1}}}{\sqrt{-2U'(x_0) - 2\frac{r(y_{i+1}+x_0)}{y_{i+1}}}}$$

Applying the ratio test to the right hand side of the above equation shows that

$$\begin{aligned}
&\lim_{i\to\infty} \frac{\sqrt{y_{i+1}}}{\sqrt{-2U'(x_0) - 2\frac{r(y_{i+1}+x_0)}{y_{i+1}}}} \cdot \frac{\sqrt{-2U'(x_0) - 2\frac{r(y_i+x_0)}{y_{i+1}}}}{\sqrt{y_i}} \\
&= \lim_{i\to\infty} \sqrt{\frac{y_{i+1}}{y_i}} \cdot \sqrt{\frac{-2U'(x_0) - 2\frac{r(y_i+x_0)}{y_i}}{-2U'(x_0) - 2\frac{r(y_{i+1}+x_0)}{y_{i+1}}}} \\
&= \sqrt{\frac{1}{2}} \sqrt{\frac{-2U'(x_0) - 2\cdot 0}{-2U'(x_0) - 2\cdot 0}} \\
&= \sqrt{\frac{1}{2}} < 1
\end{aligned}$$

so that the series converges. Thus the improper integral is bounded above and therefore also converges. Applying the same argument to the other side where $U'(x_1) > 0$ shows that the equation

$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{\sqrt{2\big(E - U(x)\big)}}$$

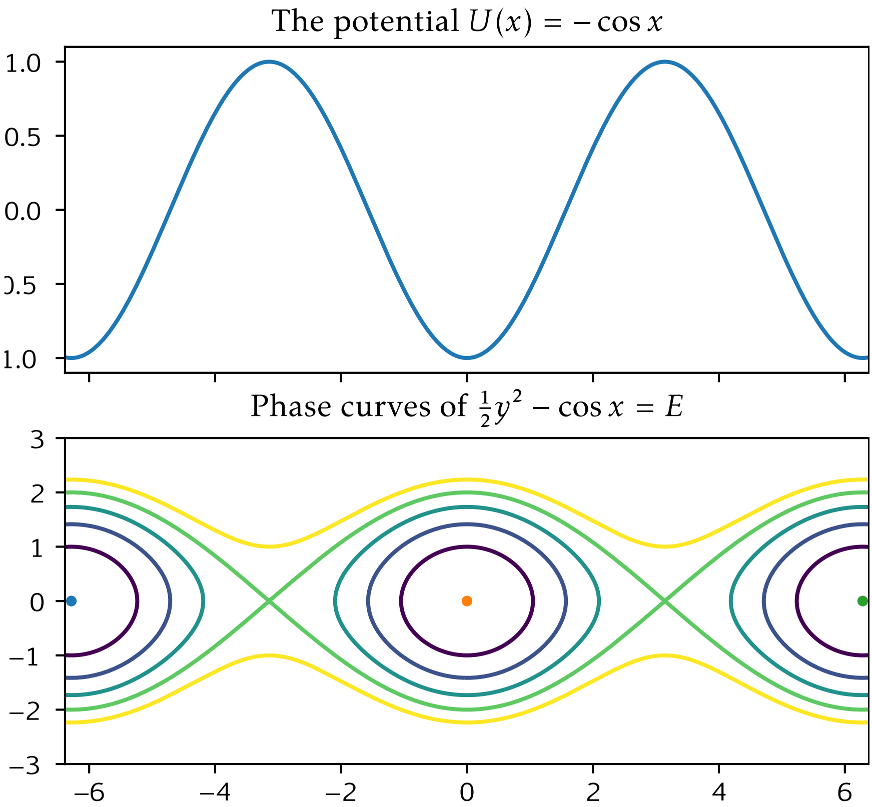

Figure 1: Problem 19-2: phase curves for $\ddot{x} = -\sin x$

applies even for $U(x_1) = U(x_2) = E$.

Exercise 19-2: Draw the phase curves for the "equation of an ideal planar pendulum": $\ddot{x} = -\sin x$.

Solution: The potential energy is $U(x) = -\cos x$ so that the phase curves are level sets of $\frac{1}{2}y^2 - \cos x = E$ for varying energy levels E. Various phase curves are shown in Figure 1.

Exercise 19-3: Draw the phase curves for the "equation of a pendulum on a rotating axis": $\ddot{x} = -\sin x + M$.

Solution: The potential energy is $U(x) = -\cos x - Mx$ so that the phase curves are level sets of $\frac{1}{2}y^2 - \cos x - Mx = E$ for varying energy levels E. Various phase curves are shown in Figure 2.

Exercise p. 19-4: Find the tangent lines to the branches of the critical level corresponding to the maximal potential energy $E = U(\xi)$.

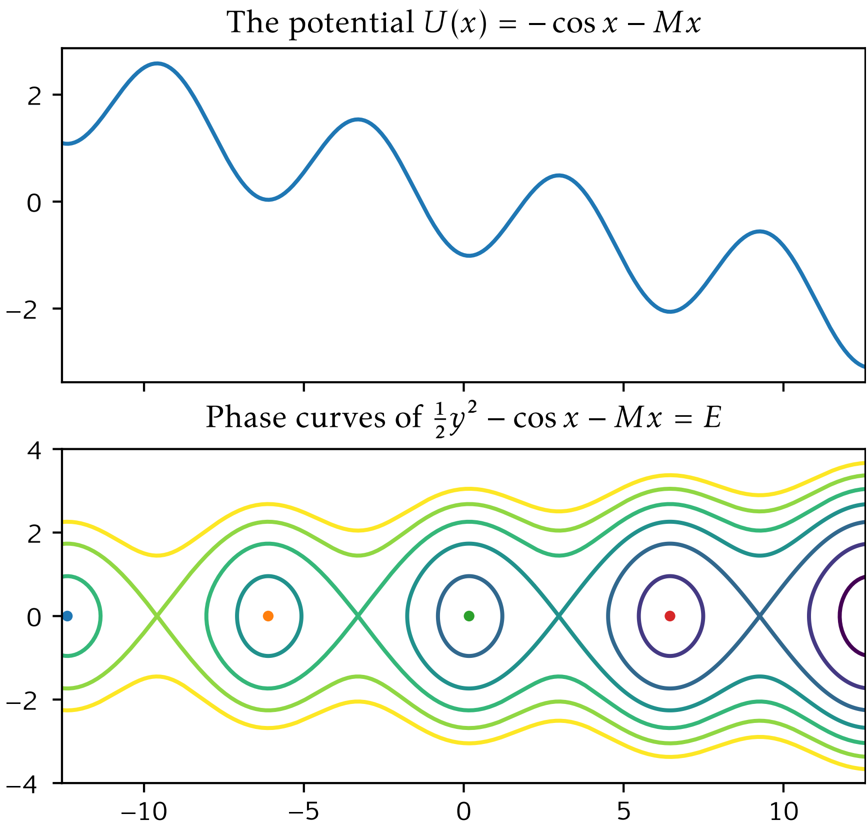

Figure 2: Problem 19-3: phase curves for $\ddot{x} = -\sin x + M$

This solution is entirely thanks to lolcats who explained it to me in the Libera chat ##math channel.

Solution: Every phase curve lies in the level set

$$\frac{1}{2}y^2 + U(x) = E$$

A critical level is such that $U(\xi) = E$ is a local maximum at ξ. Thus we also have that $U'(\xi) = 0$ and that $U''(\xi)$ is negative in a neighborhood of ξ. Taylor's theorem with remainder allows us to write U as

$$\begin{aligned} U(x) &= U(\xi) + U'(\xi)(x-\xi) + \frac{1}{2}U''(\xi)(x-\xi)^2 + r(x) \\ &= E + \frac{1}{2}U''(\xi)(x-\xi)^2 + r(x) \end{aligned}$$

where the remainder term r satisfies $\lim_{x\to\xi} \frac{r(x)}{(x-\xi)^2} = 0$). Thus we have that

$$\begin{aligned} y(x)^2 &= 2(E - U(x)) \\ &= 2(E - E - \frac{1}{2}U''(x)(x-\xi)^2 - r(x)) \\ &= -U''(x)(x-\xi)^2 - 2r(x) \end{aligned}$$

so that

$$\frac{y(x)^2}{(x-\xi)^2} = -U''(x) - 2\frac{r(x)}{(x-\xi)^2}$$

and thus

$$\frac{y(x)}{x-\xi} = \pm\sqrt{-U''(x) - 2\frac{r(x)}{(x-\xi)^2}}$$

This is the slope of the secant line from $(\xi, 0)$ to $(x, y(x))$ so that taking the limit $x \to \xi$ gives us the slope of the tangent line at ξ:

$$\lim_{x\to\xi} \frac{y(x)}{x-\xi} = \pm\sqrt{-U''(\xi)}$$

Thus we have that

$$\pm\sqrt{-U''(\xi)}(x-\xi)$$

are the tangent lines to the level curve at ξ.

Exercise p. 21-1: Show that the system with potential energy $U = -x^4$ does not define a phase flow.

Solution: Any potential phase flow g^t is composed of curves $c_p(t) = g^t p$ which satisfy the equations of motion:

$$\begin{aligned} \dot{x}_p(t) &= y_p(t) \\ \dot{y}_p(t) &= -\frac{dU}{dx}(x_p(t)) \\ &= 4\left(x_p(t)\right)^3 \end{aligned}$$

where $c_p(t) = \left(x_p(t), y_p(t)\right)$. We investigate one of these curves to show that it cannot give rise to a phase flow.

Consider the point $p = \left(1, \sqrt{2}\right)$ which satisfies $\frac{1}{2}\sqrt{2}^2 - 1^4 = 1 - 1 = 0$ thus lying on the level curve corresponding to $E = 0$. We will derive a formula for $c_p(t) = \left(x_p(t), y_p(t)\right)$. Since we have in this case that $U(x) = -x^4$, $E = 0$ and $x_p(0) = 1$, we use the result of Problem 18-4 as follows:

$$\begin{aligned}
t &= t - 0 \\
&= \int_{x_p(0)}^{x_p(t)} \frac{dx}{\sqrt{2\left(E - U(x)\right)}} \\
&= \int_{1}^{x_p(t)} \frac{dx}{\sqrt{2x^4}} \\
&= \frac{1}{\sqrt{2}} \int_{x_p(0)}^{x_p(t)} \frac{dx}{x^2} \\
&= \frac{1}{\sqrt{2}} \left. -\frac{1}{x} \right|_0^{x_p(t)} \\
&= \frac{1}{\sqrt{2}} \left(-\frac{1}{x_p(t)} + 1 \right)
\end{aligned}$$

so that

$$x_p(t) = \frac{1}{1 - \sqrt{2}t}$$

Since $y = \sqrt{2(E - U(x))} = \sqrt{2x^4}$ we have that

$$\begin{aligned}
y_p(t) &= \sqrt{2\left(\frac{1}{1 - \sqrt{2}t}\right)^4} \\
&= \frac{\sqrt{2}}{\left(1 - \sqrt{2}t\right)^2}
\end{aligned}$$

from which we can verify that

$$\begin{aligned}
\frac{d}{dt} x_p(t) &= -\frac{1}{\left(1 - \sqrt{2}t\right)^2} \cdot -\sqrt{2} \\
&= \frac{\sqrt{2}}{\left(1 - \sqrt{2}t\right)^2} \\
&= y_p(t)
\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}y_p(t) &= -\frac{2\sqrt{2}}{\left(1-\sqrt{2}t\right)^3}\cdot -\sqrt{2}\\ &= \frac{4}{\left(1-\sqrt{2}t\right)^3}\\ &= 4\left(x_p(t)\right)^3\end{aligned}$$

so that

$$c_p(t) = \left(\frac{1}{1-\sqrt{2}t}, \frac{\sqrt{2}}{\left(1-\sqrt{2}t\right)^2}\right)$$

does indeed define the unique curve at $p = \left(1, \sqrt{2}\right)$ that satisfies the equations of motion.

Since the denominators of these curves are 0 at $t = \frac{1}{\sqrt{2}}$, the maximal domain of definition of this integral curve is $\left]-\infty, \frac{1}{\sqrt{2}}\right[$ and we have that

$$\lim_{t\to\frac{1}{\sqrt{2}}} c_p(t) = (\infty, \infty)$$

Thus by fixing $t = \frac{1}{\sqrt{2}}$ and varying p we have that

$$\lim_{q\to\left(1,\sqrt{2}\right)} g^{\frac{1}{\sqrt{2}}}q = (\infty, \infty)$$

by the theorem on the smooth dependence on initial conditions, for example see Corollary 4 of Chapter 2 of Arnold, *Ordinary Differential Equations*. But then $g^{1/\sqrt{2}}$ cannot be a diffeomorphism since there is no value we can assign at $p = \left(1, \sqrt{2}\right)$ so that $g^{1/\sqrt{2}}$ will be continuous, much less smooth.

Figure 3 shows the vector field $y\frac{\partial}{\partial x} + 4x^3\frac{\partial}{\partial y}$ within the square $[-1, 1] \times [-1, 1]$, various level sets of $\frac{1}{2}y^2 - x^4 = E$, the curve $c_p(t)$ in blue and the point $p = \left(1, \sqrt{2}\right)$ at the red $\times$.

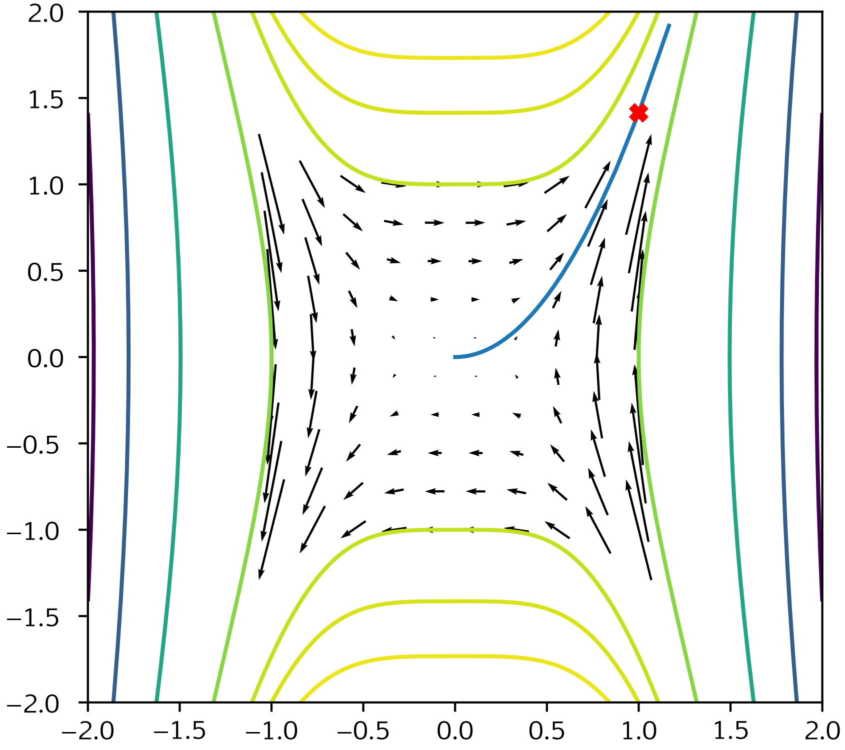

Figure 3: Problem 21-1